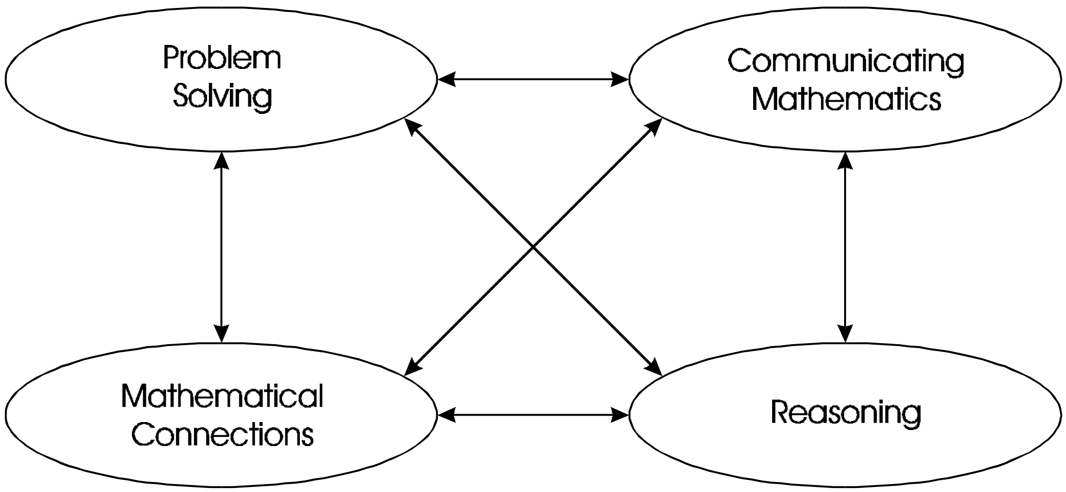
THE FIRST FOUR STANDARDS



The First Four Standards deal with mathematical processes that apply to every topic in the mathematics curriculum. Students should always be improving their ability to **solve problems**, to **communicate about mathematics**, to **make connections** within mathematics and between mathematics and other subjects, and to **reason mathematically**.

Although the First Four Standards are conceptually separate, in practice they are interwoven. For example, when students are working in groups to solve a mathematics problem that may have arisen in another area, they use communication to explain their solution strategies and reasoning to justify their conclusions. The inter-relation of the First Four Standards is reflected in the graphic that appears above and elsewhere in this chapter.

This *Framework* therefore discusses the First Four Standards in a single chapter. It begins with a separate K-12 Overview for each of the First Four Standards. Following that, as with other standards, there is a grade-level section for each of the grade levels K-2, 3-4, 5-6, 7-8, and 9-12. In the grade-level sections, the First Four Standards are discussed together in a grade-level overview, vignettes are used to illustrate how the First Four Standards can be achieved at that grade-level, and the vignettes are then related to the cumulative progress indicators for each of the First Four Standards at that grade-level.

In the *Curriculum and Evaluation Standards for School Mathematics* of the National Council of Teachers of Mathematics, these four standards are referred to as "process standards." In the *Mathematics Standards* of the New Jersey State Department of Education's *Core Curriculum Content Standards*, they are enumerated among the core curriculum content standards because, like the other content standards, the First Four Standards involve concepts and skills that students must master, and explicit focus on them is necessary in order for that mastery to be achieved.

STANDARD 1 — PROBLEM SOLVING

K-12 Overview

All students will develop the ability to pose and solve mathematical problems in mathematics, other disciplines, and everyday experiences.

Descriptive Statement

Problem posing and problem solving involve examining situations that arise in mathematics and other disciplines and in common experiences, describing these situations mathematically, formulating appropriate mathematical questions, and using a variety of strategies to find solutions. By developing their problem-solving skills, students will come to realize the potential usefulness of mathematics in their lives.

Meaning and Importance

Problem solving is a term that often means different things to different people. Sometimes it even means different things at different times for the same people! It may mean solving simple word problems that appear in standard textbooks, applying mathematics to real-world situations, solving nonroutine problems or puzzles, or creating and testing mathematical conjectures that may lead to the study of new concepts. In every case, however, problem solving involves an individual confronting a situation which she has no guaranteed way to resolve. Some tasks are problems for everyone (like finding the volume of a puddle), some are problems for virtually no one (like counting how many eggs are in a dozen), and some are problems for some people but not for others (like finding out how many balloons 4 children have if each has 3 balloons, or finding the area of a circle).

Problem solving involves far more than solving the word problems included in the students' textbooks; it is an approach to learning and doing mathematics that emphasizes questioning and figuring things out. The *Curriculum and Evaluation Standards* of the National Council of Teachers of Mathematics considers problem solving as the central focus of the mathematics curriculum.

> “As such, it is a primary goal of all mathematics instruction and an integral part of all mathematics activity. Problem solving is not a distinct topic but a process that should permeate the entire program and provide the context in which concepts and skills can be learned.” (p. 23)

Thus, problem solving involves all students a large part of the time; it is not an incidental topic stuck on at the end of the lesson or chapter, nor is it just for those who are interested in or have already mastered the day's lesson.

Students should have opportunities to pose as well as to solve problems; not all problems considered should be taken from the text or created by the teacher. However, the situations explored must be interesting, engaging, and intellectually stimulating. Worthwhile mathematical tasks are not only interesting to the students, they also develop the students' mathematical understandings and skills, stimulate them to make connections and develop a coherent framework for mathematical ideas, promote communication about mathematics, represent mathematics as an ongoing human activity, draw on their diverse background experiences and inclinations, and promote the development of all students' dispositions to do mathematics (*Professional Standards* of the National Council of Teachers of Mathematics). As a result of such activities, students come to understand mathematics and use it effectively in a variety of situations.

K-12 Development and Emphases

Much of the work that has been done in connection with problem solving stems from George Polya's book, *How to Solve It*. Polya describes four types of activities necessary for problem solving: understanding the problem, making a plan, carrying out the plan, and looking back.

The first step in solving a problem is **understanding the problem**. Suppose that we want to solve the following problem:

> *A farmer had some pigs and chickens. One day he counted 20 heads and 56 legs. How many pigs and how many chickens did he have?*

After reading the problem, we want to be sure we understand it. We might begin by noting that we probably have to use the number of heads and the number of legs in some way. We know that pigs have four legs and chickens have two. We see that there must be 20 animals in all. We might observe that, if the farmer had only chickens, there would be 40 legs. If, on the other hand, he had only pigs, there would be 80 legs.

Some techniques that may help students with this important aspect of problem solving — understanding the problem — include restating the problem in their own words, drawing a picture, or acting out the problem situation. Some teachers have students work in pairs on problems, with one student reading the problem and then, without referring to the written text, explaining what the problem is about to their partner.

A second type of activity relating to problem solving involves **making a plan**. For our pigs and chickens problem, the plan might be to make a chart that shows various combinations of 20 chickens and pigs and how many legs they have altogether. If we have too many legs, we need fewer pigs, and if we have too few legs, we need more pigs.

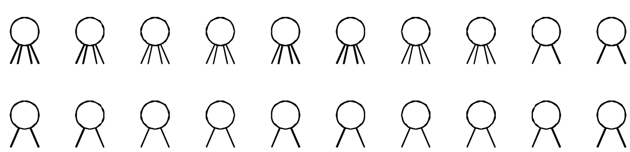
In order to be successful problem solvers, students need to become familiar with a variety of strategies that are used in making a plan for solving problems. Some of the strategies that are especially useful are making a list, making a chart or a table, drawing a diagram, making a model, simplifying the problem, looking for a pattern, using manipulatives, working backwards, eliminating possibilities, using a formula or equation, acting out the problem, using logic, using guess and check, using a spreadsheet, using a computer sketching program like *Geometer's Sketchpad, The Geometry SuperSupposer*, or *Cabri*, writing a computer program, or using a graphing calculator.

Let's **carry out our plan** for using a chart to solve the pigs and chickens problem. If we have 10 pigs (that's 40 legs) and 10 chickens (that's 20 legs), then we have 60 legs — that's too many legs. Let's try 9 pigs and 11 chickens — still too many. How about 8 pigs and 12 chickens? That's just right.

Number of Pigs	Number of Chickens	Number of Legs
10	10	40 + 20 = 60
9	11	36 + 22 = 58
8	12	32 + 24 = 56

Carrying out the plan is sometimes the easiest part of solving a problem. However, many students jump to this step too soon. Others carry out inappropriate plans, or give up too soon and stop halfway through solving the problem. To reinforce the process of making a plan and carrying it out, teachers might use the following technique: Divide a sheet of notebook paper into two columns. On the left side of the page, the student solves the problem. On the right side of the page, the student writes about what is going on in his/her mind concerning the problem. *Is the problem hard? How can you get started? What strategy might work? How did you feel about the problem?*

Let's **look back** at the problem we have just finished. The pigs and chickens problem may remind some of you of other problems you have solved; it's a little bit like some of the algebra problems involving the value of coins. Others may be intrigued by the pattern that we seem to have started in the last column of our chart and seek an explanation for this pattern. Still others may have solved this problem a completely different way; we could discuss all of the different strategies the students used and decide which ones seem most effective. One strategy used by young children is to draw a picture. Twenty circles represent the animals' heads. Each animal gets 2 legs. Additional pairs of legs are drawn on animals, starting at the left, until there are 56 legs.



This looking back activity is where students reflect upon the problem. *Does the answer make sense? Is the question answered completely? How is the problem like others you have seen? How is it different?*

While it might seem most logical to begin problem solving with Polya's first activity and proceed through each activity until the end, not all successful problem solvers do so. Many successful problem solvers begin by understanding the problem and making a plan. But then as they start carrying out their plan, they may find that they have not completely understood the problem, in which case they go back to step one. Or they may find that their original plan is extremely difficult to pursue, so they go back to step two and select another approach. By using these four activities as a general guide, however, students can become more adept at monitoring their own thinking. This "thinking about their thinking" can help them to improve their problem solving skills.

Students move through a continuum of stages in their development as problem solvers (Kantowski, 1980). Initially, they have little or no understanding of what problem solving is, of what a strategy is, or of the mathematical structure of a problem. Such students usually do not know where to begin to solve a problem; the teacher must model the problem solving process for these students. At the second level, students are able to follow someone else's solution and may suggest strategies for similar problems. They may participate actively in group problem solving situations but feel insecure about independent activities, requiring the teacher's continued support. At the third level, students begin to be comfortable with solving problems, suggesting strategies different from those they have seen used before. They understand and appreciate that problems may have multiple solutions or perhaps even no solution at all. Finally, at the last level, students are not only adept at solving problems, they are also interested in finding elegant and efficient solutions and in exploring alternate solutions to the same problem. In teaching problem solving, it is important to address the needs of students at each of these levels within the classroom.

IN SUMMARY, the real test of whether a student knows mathematics is whether she can use it in a problem situation. Students should experience problems as introductions to learning about new topics, as applications of content already studied, as puzzles or non-routine problems that have many solutions, and as situations that have no one best answer. They should not only solve problems but also pose them. They should focus on understanding a problem, making a plan for solving it, carrying out their plan, and then looking back at what they have done.

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Software

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The Geometry SuperSupposer. Sunburst Communications.

STANDARD 2 — COMMUNICATION

K-12 Overview

All students will communicate mathematically through written, oral, symbolic, and visual forms of expression.

Descriptive Statement

Communication of mathematical ideas will help students clarify and solidify their understanding of mathematics. By sharing their mathematical understandings in written and oral form with their classmates, teachers, and parents, students develop confidence in themselves as mathematics learners and enable teachers to better monitor their progress.

Meaning and Importance

Mathematics can be thought of as a language that must be meaningful if students are to communicate mathematically and apply mathematics productively. Communication plays an important role in making mathematics meaningful; it enables students to construct links between their informal, intuitive notions and the abstract language and symbolism of mathematics. It also plays a key role in helping students make critical connections among physical, pictorial, graphic, symbolic, verbal, and mental representations of mathematical ideas. When students see that one representation, such as an equation, can describe many situations, they begin to understand the power of mathematics. When they realize that some ways of representing a problem are more helpful than others, they begin to understand the flexibility and usefulness of mathematics.

K-12 Development and Emphases

Communication involves a **variety of modes**: speaking, listening, writing, reading, and representing visually (with pictures, graphs, diagrams, videos, or other visual means). Each of these can help students understand mathematics and use it effectively. Students should also use communication to **generate and share ideas.** Communicating with each other, with peers, with parents, with other adults, and with the teacher, orally and in writing, helps students learn mathematics as they clarify their own ideas and listen to those of others. The language of mathematics itself is a thinking tool that facilitates mathematical understanding and connects to natural language and everyday thinking.

Students need to have many experiences in communicating about mathematics in a **variety of settings**. Some experiences will involve working in pairs; for example, kindergartners can sit back-to-back with one giving the other directions about how to make a tower of Unifix cubes. Other experiences will involve working in small groups, such as when tenth-graders combine information from several separate clues to find the distance around a park. Some experiences will involve explaining something to the whole class, while others may involve drawing a picture, making a model, or writing in a journal.

Students need to learn the appropriate **use of mathematical language and symbols**. Most experiences relating to mathematical communication will involve the use of natural language, but some will also involve the use of tables, charts, graphs, manipulatives, equations, computers, and calculators. Students should not only be able to use each of these different media to describe mathematical ideas and solutions to problems, but they should also be able to interrelate the descriptions obtained using different media.

IN SUMMARY, communicating mathematics — orally, in writing, and using symbols and visual representations — is vitally important to learning and using mathematics. Students should use a variety of forms of communication in a variety of settings to generate and share ideas.

STANDARD 3 — MATHEMATICAL CONNECTIONS

K-12 Overview

> All students will connect mathematics to other learning by understanding the interrelationships of mathematical ideas and the roles that mathematics and mathematical modeling play in other disciplines and in life.

Descriptive Statement

Making connections enables students to see relationships between different topics and to draw on those relationships in future study. This applies within mathematics, so that students can translate readily between fractions and decimals, or between algebra and geometry; to other content areas, so that students understand how mathematics is used in the sciences, the social sciences, and the arts; and to the everyday world, so that students can connect school mathematics to daily life.

Meaning and Importance

Although it is often necessary to teach specific concepts and procedures, mathematics must be approached as a whole; concepts, procedures, and intellectual processes are interrelated. More generally, although students need to learn different content areas, they also should come to see all learning as interwoven. In a very real sense, the whole is greater than the sum of its parts. Thus, the curriculum should include deliberate efforts, through specific instructional activities, to connect ideas and procedures both among different mathematical topics and with other content areas.

K-12 Development and Emphases

One important focus of this standard is that of **unifying mathematical ideas** — major mathematical themes which are relevant in several different strands. They emerge when a higher-level view of content is taken. They tie together individual mathematical topics, revealing general principles at work in several different strands and showing how they are related. Unifying ideas also set priorities, and the curriculum should be designed so that students develop depth of understanding in each of the unifying ideas at their grade level. Since it often takes years to achieve understanding of these unifying ideas, students will encounter them repeatedly in many different contexts.

An example of a unifying idea is the concept of proportional relationships. Proportional relationships play a key role in a wide variety of important topics, such as ratios, proportions, rates, percent, scale, similar geometric figures, slope, linear functions, parts of a whole, probability and odds, frequency distributions and statistics, motion at constant speed, simple interest, and comparison. Awareness of the common principle operating in all these topics is an important part of mathematical understanding. Unifying ideas for grades K-4 include quantification (*how much? how many?*), patterns (finding, making, and describing), and representing quantities and shapes. Unifying ideas for the middle grades include proportional relationships, multiple representations, and patterns and generalization. For the high school, the unifying ideas are mathematical modeling, functions and variation (*how is a change in one thing associated with a change in another?*), algorithmic thinking (developing, interpreting, and analyzing mathematical procedures), mathematical argumentation, and a continued focus on multiple representations.

Another focus of this standard is **mathematical modeling**, developing mathematical descriptions of real-world situations and (usually) predicting outcomes based on that model. Even very young children use mathematics to model real situations, counting candies or cookies or matching cups and saucers on a one-to-one basis. Affirming the ways things make sense outside of school and connecting them to things in school is important for students. Older students use mathematical modeling as they develop the concepts of function and variable, extending ideas they already have about growth, motion, or cause and effect. At various grade levels, students can use their experiences waiting in lines to investigate the general relationships among waiting time, the number of people in the line, the position at the end of the line, and the length of time each person takes to buy a ticket or a lunch. Older students can further develop a mathematical model that can help them predict waiting time and formulate and evaluate their suggestions for solving a real problem — getting everyone through the school lunch line more quickly.

Throughout the grades, students need to develop their understanding of the **relationship of mathematics to other disciplines.** Mathematics is frequently used as a tool in other disciplines. In science, students measure quantities and analyze data. In social studies, they collect and analyze data and make choices using discrete mathematics. In art, they generate designs and show perspective. Other disciplines can also provide interesting contexts to learn about new mathematical ideas. For example, students might learn about symmetry by generating symmetric designs with paint, or they might explore exponential functions by modeling a dying population by repeatedly scattering M&Ms, at each step removing those that have the M showing, so that about half of the population "dies" each time.

Every person views the process of making connections between disciplines differently, and everyone works in different circumstances in which often-competing goals must be balanced. Nevertheless, it is important to establish some means of making explicit the connections between mathematics and other disciplines. There are several ways in which the presentation of content can be approached, ranging from "fragmented" to "integrated" (adapted from Fogarty, 1991):

1. Instruction may be *fragmented*. This is the traditional model of separate and distinct disciplines, each of which is presented in isolation.
2. Each subject may be *connected* within itself; concepts are explicitly connected within each course and from course to course within each subject area, but connections between subjects are not made.
3. Instruction in each subject may have *nested* within it discussion of particular topics, but connections are not made across the disciplines.
4. Teachers may arrange and *sequence* related topics or units of study to coincide with one another. Similar ideas are taught in concert but remain unconnected.
5. *Shared* planning and teaching take place; overlapping concepts or ideas emerge as organizing elements.

6. A fertile theme is *webbed* to each subject area; teachers use the theme to sift out appropriate concepts, topics, and ideas.
7. The *threaded* approach weaves shared topics throughout each subject.
8. The *integrated* approach involves focusing on overlapping topics and concepts, and includes team teaching.

While the first two approaches do not provide for the establishment of appropriate connections between subjects, the remaining six do provide some degree of interaction. Teachers must consider carefully which of these is most appropriate and feasible for their own situation.

This discussion addresses the connections between mathematics and other disciplines. Of special significance because of their many commonalities is **the relationship of mathematics to science**. Berlin and White (1993) have identified six areas in which mathematics and sciences share concepts or skills: ways of learning, ways of knowing, process and thinking skills, conceptual knowledge, attitudes and perceptions, and teaching strategies.

Ways of Learning — The two disciplines have a common perspective on how students experience, organize, and think about science and mathematics. Both disciplines endorse active, exploratory learning with opportunities for students to share and discuss ideas. Students must *do* science and mathematics in order to learn science and mathematics.

Ways of Knowing — Both science and mathematics use patterns to help students develop understanding. Even very young children seek to make sense of patterns in order to make sense of their world. They make generalizations based on what they have observed and apply these generalizations to new situations. Sometimes their guess works, reinforcing the generalization, and sometimes it doesn't, requiring the child to revise the generalization.

An example from chemistry demonstrates the similar ways of knowing used in mathematics and science. Chemist Dmitri Mendeleev proposed a periodic table of the elements based on increasing atomic weights. Sometimes he left open spaces in the table, where he reasoned that unknown elements should go. In 1869, when he arranged his table, the element gallium was unknown; however, Mendeleev predicted its existence. He based his predictions on the properties of aluminum (which appeared directly above gallium in the table). Mendeleev even went so far as to predict the melting point, boiling point, and atomic weight of the then-unknown gallium, which he called eka-aluminum. Six years later, while analyzing zinc ore, the French chemist Lecoq de Boisdaudran discovered the element gallium. Its properties were almost identical to those Mendeleev had predicted.

Process and Thinking Skills — Central to both disciplines are process skills. Mathematics focuses primarily on the following four process skills: problem solving, reasoning, communication, and connections. Basic process skills in science (Tobin and Capie, 1980) include observing, inferring, measuring, communicating, classifying, formulating hypotheses, experimenting, interpreting data, and formulating models.

Conceptual Knowledge — There is considerable overlapping of content between science and mathematics. By examining the concepts, principles, and theories of science and mathematics, those ideas that are unique to one subject and those which overlap both disciplines can be identified. Some of the “big ideas” which are common to both include conservation (of number, volume, etc.), equilibrium, measurement, models (including both concrete and symbolic), patterns (including trends, cycles, and chaos), probability and statistics, reflection, scale (including size, duration, and speed), symmetry, systems, variables, and vectors. An example of a way to interrelate science and mathematics contents is to link population dynamics and genetics in science with sampling and probability in mathematics.

Attitudes and Perceptions — Mathematics and science share certain values, attitudes, and ways of thinking: accepting the changing nature of science and mathematics, basing decisions and actions on data, exhibiting a desire for knowledge, having a healthy degree of skepticism, relying on logical reasoning, being willing to consider other explanations, respecting reason, viewing information in an objective and unbiased manner, and working together cooperatively to achieve better understanding. Both disciplines also value flexibility, initiative, risk-taking, curiosity, leadership, honesty, originality, inventiveness, creativity, persistence, and resourcefulness, as well as being thorough, careful, organized, self-confident, self-directed, and introspective, and valuing science and mathematics (*Science for All Americans*). Students' engagement in personal and social issues and interests may also help to encourage, support, and nurture their confidence in their ability to do science and mathematics.

Teaching Strategies — The shared goals of mathematics and science instructions are, according to *Science for All Americans*, to have students acquire scientific and mathematical knowledge of the world as well as scientific and mathematical habits of mind. Both disciplines support teaching strategies which foster inquiry and problem-solving, promote discourse among students, challenge students to take responsibility for their own learning and to work collaboratively, encourage all students to participate fully, and nurture a community of learners (*National Science Education Standards*).

IN SUMMARY, making connections within mathematics and between mathematics and other subjects not only helps students understand the mathematical ideas more clearly, it also captures their interest and demonstrates how mathematics is used in the real world. Important connections that need to be established include working with unifying mathematical themes, using mathematical modeling, and relating mathematics to other disciplines and to the real world.

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STANDARD 4 — REASONING

K-12 Overview

> All students will develop reasoning ability and will become self-reliant, independent mathematical thinkers.

Descriptive Statement

Mathematical reasoning is the critical skill that enables a student to make use of all other mathematical skills. With the development of mathematical reasoning, students recognize that mathematics makes sense and can be understood. They learn how to evaluate situations, select problem-solving strategies, draw logical conclusions, develop and describe solutions, and recognize how those solutions can be applied. Mathematical reasoners are able to reflect on solutions to problems and determine whether or not they make sense. They appreciate the pervasive use and power of reasoning as a part of mathematics.

Meaning and Importance

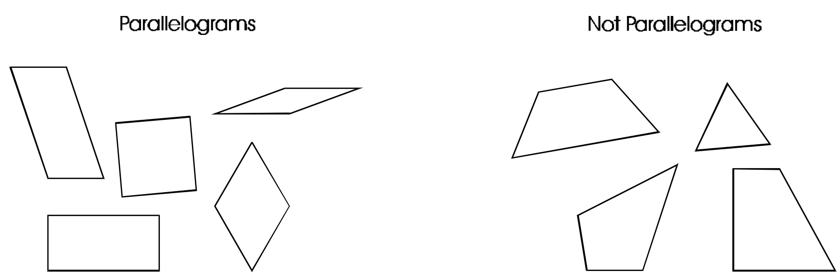
There are various terms used to refer to "reasoning": critical thinking, higher-order thinking, logical reasoning, or simply reasoning. Different subject areas tend to use different terms. Across all of these subject areas, however, there are commonalities. The following phrases often appear in discussions of how reasoning is used (adapted from Resnick, 1987, pp 2-3):

- *Nonalgorithmic* — The route to a solution is not fully specified in advance.
- *Complex* — The complete path to a solution is not fully apparent from any single vantage point.
- *Multiple criteria* — The conditions established in the problem may conflict with one another.
- *Uncertainty* — Not everything that bears on the task at hand is known.
- *Imposing meaning* — The individual must find structure in apparent disorder.
- *Effortful* — There is considerable mental work involved in the elaborations and judgments required.
- *Self-regulation* — The individual monitors his or her own progress, and determines the appropriate course of action.
- *Multiple solutions* — There is no single "best" solution; rather, there are many solutions, each with costs and benefits.
- *Nuanced judgment* — The results must be interpreted.

K-12 Development and Emphases

Every student has potential for higher-order thinking. The key is to unlock the world of mathematics through a student's natural inclination to strive for purpose and meaning. Reasoning is fundamental to the knowing and doing of mathematics. Conjecturing and demonstrating the logical validity of conjectures are the essence of the creative act of *doing* mathematics. To give more students access to mathematics as a powerful way of making sense of the world, it is essential that an emphasis on reasoning pervade all mathematical activity. In order to become confident, self-reliant mathematical thinkers, students need to develop the capability to confront a mathematical problem, persevere in its solution, and evaluate and justify their results.

Inductive reasoning involves looking for patterns and making generalizations. For example, students use this type of reasoning when they look at many different parallelograms, and try to list the characteristics they have in common. The reasoning process is enhanced by also considering figures that are not parallelograms and discussing how they are different.



Students may use inductive reasoning to discover patterns in multiplying by ten or a hundred or in working with exponents. Learning mathematics should involve a constant search for patterns, with students making educated guesses, testing them, and then making generalizations.

Many students use inductive reasoning more frequently than teachers realize, but the generalizations that they form are not always correct. For example, a student may see the examples 16/64 = 1/4 and 19/95 = 1/5 and reason inductively that the common digits in a fraction may be canceled. The student must realize that she needs to continue to test her conjecture before making such a generalization, since 17/76 ≠ 1/6, for example. Students must also realize that while inductive reasoning demonstrates the power of mathematics and allows great leaps forward in understanding, it is insufficient in itself. The generalizations that are obtained by using inductive reasoning can only be accepted by “proving” them through deductive reasoning.

Deductive reasoning involves making a logical argument, drawing conclusions, and applying generalizations to specific situations. For example, once students have developed an understanding of “parallelogram,” they apply that generalization to new figures to decide whether or not each is a parallelogram. This kind of reasoning also may involve eliminating unreasonable possibilities and justifying answers. Although students as young as first graders can recognize valid conclusions, the ability to use deductive reasoning improves as students grow older. More complex reasoning skills, such as recognizing invalid arguments, are appropriate at the secondary level.

Understanding the power of reasoning to make sense of mathematics is critical to helping students become self-reliant, independent mathematical thinkers. Students must be able to judge for themselves the accuracy of their answers; they must be able to apply mathematical reasoning skills in other subject areas and in their daily lives. They must recognize that mathematical reasoning can be used in many different situations to help them make choices and reach decisions.

IN SUMMARY, mathematical reasoning is the glue that binds together all other mathematical skills. By using inductive and deductive reasoning as they learn mathematical concepts and solve mathematical problems, students come to recognize the extent to which reasoning applies to mathematics and to their world.

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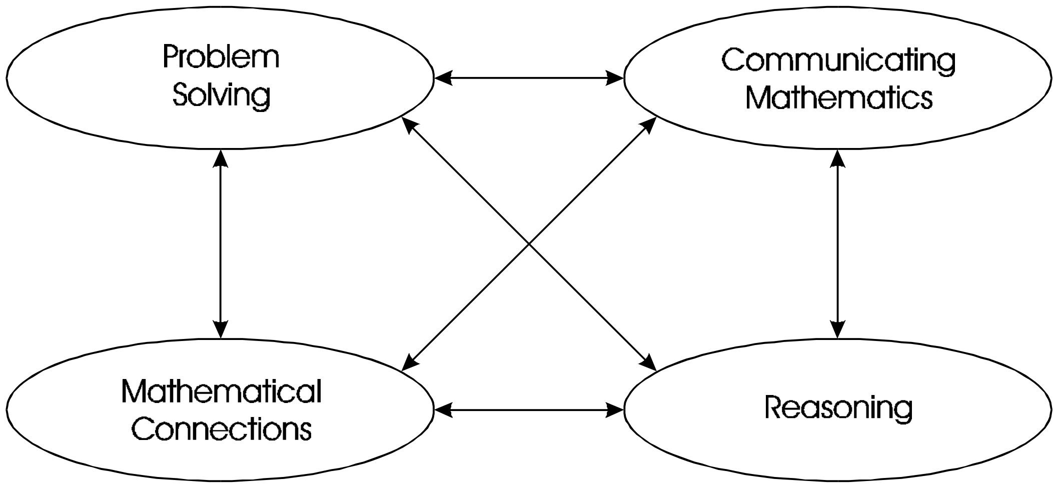
The First Four Standards — Grades K-2



Overview

Young children enter school with informal strategies for solving mathematical problems, communication skills, ideas about how number and shape connect to each other and to their world, and reasoning skills. In grades K-2, students should build upon these informal strategies.

Early instruction in **problem solving** should focus on taking time to understand the problem before rushing to solve it. Kindergartners should begin, for example, by representing problems using physical objects. By second grade, students should begin to move away from dependence on physical objects towards the use of pictures and figures. One of the goals of problem solving in numerical situations is to move students toward the use of more efficient problem solving strategies — from modeling with concrete objects to counting methods to using number facts. Even kindergartners should have experience with multiple-step problems (*Mary has 3 cookies. She eats one. Her mother gives her two more. How many cookies does she have now?*) in order to focus their attention on understanding the problem and developing a plan for its solution. Students should be able to describe how they have solved a problem and justify their answer. They should also develop the habit of comparing problems to each other, noting how they are alike and different.

Communication activities in grades K-2, whether with individuals, small groups, or the whole class, initially emphasize oral (e.g., counting) and pictorial representations. Much time is spent, however, in introducing students to symbolic representations (e.g., numerals and symbols for operations). As students develop written communication skills, they also begin to communicate in writing about mathematics. At first, the teacher may write the students' responses on the board or on sentence strips in order to facilitate this written communication. Students use many concrete representations (e.g., base ten blocks, pattern blocks) and need to learn how to represent their work with these manipulatives through pictures. Students also begin to communicate mathematics using graphs and diagrams.

Many **mathematical connections** begin to be established in kindergarten. Students should connect the number three to triangles, for example, as well as to sets of three objects and the numeral 3. Especially important are quantification (*how much? how many?*), patterns, and representing quantities and shapes. Using children's literature to motivate and set a context for problem solving and learning mathematics is especially appropriate for K-2, as is illustrated in one of the following vignettes. Connections to social studies may involve using graphs to describe characteristics of the class, the school, or the community.

Many connections between science and mathematics can be established, from looking for patterns to developing specific skills in measurement and data collection. Children observe life cycles and cycles in nature, such as the seasons, and the growth and decay of plant forms. Children begin by using words to describe physical characteristics: color intensity (bright or dull), sound volume (loud or quiet), temperature (hot or cold), and size (longest or shortest). This allows them to make simple descriptive comparisons and to place objects in an order. They move on to using numbers to describe such characteristics. For example, students might measure the height of plants at different times, summarize their data in a table, and prepare a graph (bar or line) showing the height over time. They might repeat the experiment with different growing conditions, and then compare their graphs for the different conditions.

Students in grades K-2 should spend a great deal of time on inductive **reasoning**, looking for patterns, making educated guesses, generating hypotheses, and forming generalizations based on their experiences. They should also begin to develop some skill in drawing logical conclusions and justifying answers (deductive reasoning), perhaps by using manipulatives such as attribute blocks. They should continually strive to make sense of mathematics by using reasoning to predict answers and compare and contrast examples and problem situations.

In grades K-2, students build on what they already know as they develop their skills in problem solving, communication, mathematical connections, and reasoning. They begin to move from informal, intuitive strategies and processes towards more symbolic representations and more explicit recognition of their thinking strategies.

On-Line Resources

http://dimacs.rutgers.edu/nj_math_coalition/framework.html/

The *Framework* will be available at this site during Spring 1997. In time, we hope to post additional resources relating to this standard, such as grade-specific activities submitted by New Jersey teachers, and to provide a forum to discuss the *Mathematics Standards*.

The First Four Standards — Grades K-2

Vignette — Will a Dinosaur Fit?

Standards: In addition to the First Four Standards, this vignette highlights Standards 6 (Number Sense), 7 (Geometry), 9 (Measurement), and 11 (Estimation).

The problem: The second grade was in the midst of a unit on dinosaurs when the teacher read to her class the book *Danny and the Dinosaur* by Syd Hoff (Harper & Row, 1958). After the first reading, the children re-examined some of the illustrations. One picture depicted the dinosaur larger than a block of homes, another showed the dinosaur almost completely hidden by one house. One picture showed the dinosaur taller than an apartment building and yet another showed the dinosaur not quite as tall as a lamp post. Students were intrigued by the idea that Danny's dinosaur friend did not seem to be of a consistent size. They voiced opinions about the dinosaur's actual size. Since students seemed to have a sustained interest in exploring the sizes of dinosaurs, the teacher presented students with this question: *Do you think that a dinosaur could fit into our classroom?*

The discussion: Brainstorming was encouraged by the teacher as questions such as the following were posed by students and by the teacher. *What does it mean to "fit" in the classroom? What information would we need to get in order to determine if a dinosaur could fit in our classroom? Do you think all of our answers will be the same? Why? What do we know already that might help us? What materials do you think we would need?*

Solving the problem: Students worked in groups of 3 over a period of several days. They began by choosing a specific dinosaur and then they used a variety of books and computer software in the classroom to find the size of their dinosaur. They determined the size of the classroom, choosing to measure with a trundle wheel or a tape, or by using estimation. Then they decided, by comparing the measures found in books with those made of the classroom, whether the dinosaur would fit into the classroom. Each group was responsible for creating a display and making a presentation to the class to answer the question. The displays made use of models, pictures, and text. Students with more than a few sentences to write were encouraged to make use of the word processor available in the classroom.

Summary: Students used their displays to make presentations to the class. There were a variety of answers. Those who had chosen one of the smaller dinosaurs, the velociraptors, for example, found that the dinosaur could walk through the doorway and several dinosaurs would fit in the room. Others, who had chosen larger dinosaurs, the stegosaurus, for example, found that if the dinosaur could have gotten through the doorway, several would have fit in the room. Still others, who had chosen very large dinosaurs, the brachiosaurus, for example, found that the dinosaur would not have fit into the room at all. As the presentations ended, several children suggested further explorations that might be interesting: *Would the dinosaur I chose fit into the multi-purpose room? Was the dinosaur I chose as long as the driveway in front of the school? Was the dinosaur I chose taller than the school building?*

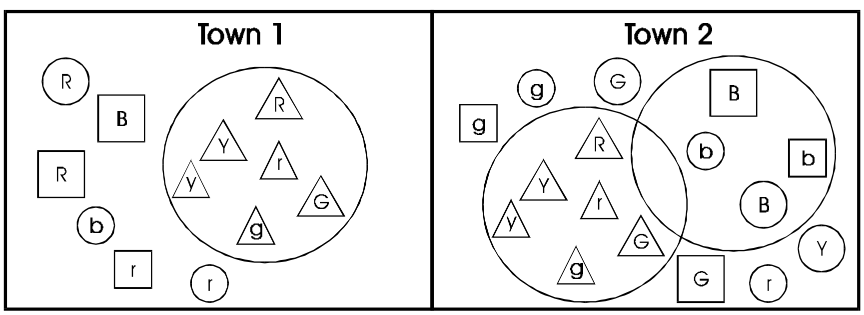
The First Four Standards — Grades K-2

Vignette — Shapetown

Standards: In addition to the First Four Standards, this vignette highlights Standards 7 (Geometry), 11 (Patterns), and 14 (Discrete Mathematics).

The problem: The students in kindergarten had been involved in a unit that allowed them to explore their town. They had been exposed to a variety of activities, including building symmetric and non-symmetric block buildings, drawing neighborhood maps, and using letter-number ordered pairs (like A-2) to locate places on a grid. In this lesson, pairs of students were challenged to build towns with attribute blocks and loops based on a rule or pattern that they made up.

The discussion: With the class sitting on the carpet in a circle, the teacher placed a loop within everyone's sight. She explained that the loop was a town and that the blocks were buildings. Using blocks of different colors, she then placed several triangles inside the loop and several non-triangles outside the loop. Ideas about the rule used to build Town 1 were discussed: *Tell me about the town. Describe a pattern that you see. Put this triangle on the carpet to follow the pattern. Put this circle on the carpet to follow the pattern. How could you tell someone else about our town so they could build one just like it?* The verbalization was then called *the rule* for the town. Town 2 was created with two loops, blocks were placed inside and outside these loops, and similar questions were raised and discussed. Several reasonable rules were suggested. For example, one rule was: triangles in one loop, blue blocks in the other loop, other colors and shapes in the overlapping loop and outside the loop. Another rule was: triangles in one loop, blue blocks in the other loop, blue triangles in the overlapping loop, and all other blocks outside the loops.



Solving the problem: Students were given loops and some attribute blocks. They were challenged to work together to build a town that used a rule. At the end of the working time, each pair of students challenged the class to place other blocks in their town and then to verbalize the rule that was used to create the town.

Summary: Students worked independently to record their town designs using crayons and shapes cut from colored construction paper. Students described the rules that they used to build their towns.

The First Four Standards — Grades K-2

Indicators

The cumulative progress indicators for grade 4 for each of the First Four Standards appear in boldface type below the standard. Each indicator is followed by a brief discussion of how the preceding grade-level vignettes might address the indicator in the classroom in kindergarten and in grades 1 and 2. The Introduction to this *Framework* contains three vignettes describing lessons for grades K-4 which also illustrate the indicators for the First Four Standards; these are entitled *Elevens Alive!*, *Product and Process*, and *Sharing a Snack*.

> Standard 1. All students will develop the ability to pose and solve mathematical problems in mathematics, other disciplines, and everyday experiences.

Experiences will be such that all students in grades K-2:

1. **Use discovery-oriented, inquiry-based, and problem-centered approaches to investigate and understand mathematical content appropriate to the early elementary grades.**
   - *Will a Dinosaur Fit?* uses the question *Do you think a dinosaur would fit into our classroom?* to launch an investigation involving measurement, geometry, estimation, and large numbers. *Shapetown* develops students' logical (deductive) reasoning skills using shapes (geometry), sorting (discrete mathematics), and pattern analysis.

2. **Recognize, formulate, and solve problems arising from mathematical situations and everyday experiences.**
   - In *Will a Dinosaur Fit?*, students recognize and help to formulate the question they will investigate, based on a book they have read and its illustrations. In *Shapetown*, students develop their own logic problems in connection with a unit in social studies on their community.

3. **Construct and use concrete, pictorial, symbolic, and graphical models to represent problem situations.**
   - Students in *Will a Dinosaur Fit?* begin with a pictorial model (the pictures in the book) and then use numerical models and graphs to represent the problem situation. Students in *Shapetown* use concrete materials (attribute blocks) to represent their problem situation and then record their "rules" using pictures.

4. **Pose, explore, and solve a variety of problems, including non-routine problems and open-ended problems with several solutions and/or solution strategies.**
   - In *Will a Dinosaur Fit?*, each group investigates a different dinosaur, using their own strategies. Different groups have different answers, depending on the size of their dinosaur.

In *Shapetown*, pairs of students pose their own problems for the others to solve.

5. **Construct, explain, justify, and apply a variety of problem-solving strategies in both cooperative and independent learning environments.**

   - The students in *Will a Dinosaur Fit?* work in groups of three, measuring the classroom and collecting data from books. The students in *Shapetown* work in pairs to develop the rules for their towns.

6. **Verify the correctness and reasonableness of results and interpret them in the context of the problems being solved.**

   - Students in *Will a Dinosaur Fit?* present their results to the class for verification. The students in *Shapetown* verify their results by having other students solve their problems.

7. **Know when to select and how to use grade-appropriate mathematical tools and methods (including manipulatives, calculators and computers, as well as mental math and paper-and-pencil techniques) as a natural and routine part of the problem-solving process.**

   - In *Will a Dinosaur Fit?* students select their own measuring tools and some use computers. They decide whether to estimate or measure and how to determine their answers (compare numbers or subtract mentally or with a calculator or with paper-and-pencil). The students in *Shapetown* use manipulatives (attribute blocks) to develop their rules.

8. **Determine, collect, organize, and analyze data needed to solve problems.**

   - The students in *Will a Dinosaur Fit?* determine what information they need to know about their dinosaurs, collect that information, organize it and analyze it. The students in *Shapetown* organize and analyze the placement of objects in the town in accordance with the rules they were given and the rules they generated or discovered.

9. **Recognize that there may be multiple ways to solve a problem.**

   - In their sharing, the students in *Will a Dinosaur Fit?* find out about the many different ways in which students address this problem. The students in *Shapetown* might explain how they figure out the "rules" their classmates use for their own towns.

Standard 2. All students will communicate mathematically through written, oral, symbolic, and visual forms of expression.

Experiences will be such that all student in grades K-2:

1. **Discuss, listen, represent, read, and write as vital activities in their learning and use of mathematics.**

   - In *Will a Dinosaur Fit?,* the students read a story, read information from books about their dinosaurs, represent their results using symbols and words, and explain their results orally. In *Shapetown*, the students listen to the teacher explain how to develop a "rule," discuss their rules in pairs as they develop them, and record their rules with a picture.

2. **Identify and explain key mathematical concepts, and model situations using oral, written, concrete, pictorial, and graphical methods.**
   - The students in *Will a Dinosaur Fit?* model their problem situations using oral and written language. Some groups may also use pictorial and/or graphical methods. The students in *Shapetown* use concrete materials to model their problems and oral methods to solve them.

3. **Represent and communicate mathematical ideas through the use of learning tools such as calculators, computers, and manipulatives.**
   - Some students in *Will a Dinosaur Fit?* use computers; others use trundle wheels or measuring tape. Students in *Shapetown* use manipulatives (attribute blocks).

4. **Engage in mathematical brainstorming and discussions by asking questions, making conjectures, and suggesting strategies for solving problems.**
   - The teacher in *Will a Dinosaur Fit?* begins the discussion of the problem by having students brainstorm what it means for a dinosaur to fit in the classroom. The students in *Shapetown* discuss the problems posed by the teacher and make conjectures as they try to solve them.

5. **Explain their own mathematical work to others, and justify their reasoning and conclusions.**
   - Students in *Will a Dinosaur Fit?* explain their work and justify their reasoning about their group's dinosaur. Students in *Shapetown* explain their work and justify their results as they challenge each other to solve their problem.

Standard 3. All students will connect mathematics to other learning by understanding the interrelationships of mathematical ideas and the roles that mathematics and mathematical modeling play in other disciplines and in life.

Experiences will be such that all students in grades K-2:

1. **View mathematics as an integrated whole rather than as a series of disconnected topics and rules**.
   - In both vignettes, the students are investigating problems that involve several content standards.

2. **Relate mathematical procedures to their underlying concepts.**
   - In *Will a Dinosaur Fit?*, students research the size of their dinosaurs, determine the size of their classroom by measuring, and compare the measures to see which is larger. In *Shapetown*, students apply the fundamental concepts of Venn diagrams.

3. **Use models, calculators, and other mathematical tools to demonstrate the connections among various equivalent graphical, concrete, and verbal representations of mathematical concepts.**

    - In *Will a Dinosaur Fit?*, students create a display and make a presentation to the class to support their conclusion. In *Shapetown*, the students verbalize the rule used for their town and then create an equivalent representation for their attribute block models using a picture.

4. **Explore problems and describe and confirm results using various representations.**

    - The second-graders in *Will a Dinosaur Fit?* use a variety of representations (symbols and words) to record their results as they investigate the problem. The students in *Shapetown* use a pictorial representation to describe their results.

5. **Use one mathematical idea to extend understanding of another.**

    - The teacher in *Will a Dinosaur Fit?* uses the students' understanding of relative size to extend their understanding of estimation and measurement. The students in *Shapetown* use their understanding of geometric shapes to build their "rules" as they learn more about logical reasoning.

6. **Recognize the connections between mathematics and other disciplines, and apply mathematical thinking and problem solving in those areas.**

    - The dinosaur lesson involves applying mathematics to learn about dinosaurs (science). The *Shapetown* lesson builds upon a social studies unit in which students use mathematics to locate buildings, construct buildings, and draw maps.

7. **Recognize the role of mathematics in their daily lives and in society.**

    - The students in *Will a Dinosaur Fit?* learn how mathematics is involved in the sizes of illustrations in the books that they read. The *Shapetown* students learn how mathematics is used in buildings, in determining locations, and in classifying and characterizing objects.

Standard 4. All students will develop reasoning ability and will become self-reliant, independent mathematical thinkers.

Experiences will be such that all students in grades K-2:

1. **Make educated guesses and test them for correctness.**

    - The students in *Will a Dinosaur Fit?* could address this indicator by predicting whether their dinosaur will fit before measuring the classroom. The students in *Shapetown* are challenged to guess the rule for placing blocks on the carpet, and then to verbalize the rule they think is being used.

2. **Draw logical conclusions and make generalizations.**

    - The students in *Will a Dinosaur Fit?* draw conclusions from the data they collect by measuring and using texts or the computer. They might also make some generalizations about dinosaurs collectively after discussing the results of all the groups. Drawing logical conclusions is the major focus of the *Shapetown* lesson.

3. Use models, known facts, properties, and relationships to explain their thinking.

- The students in *Will a Dinosaur Fit?* use models, known facts (from books and software), and relationships to explain how they know whether their dinosaur will fit. The *Shapetown* students use models to explain their thinking.

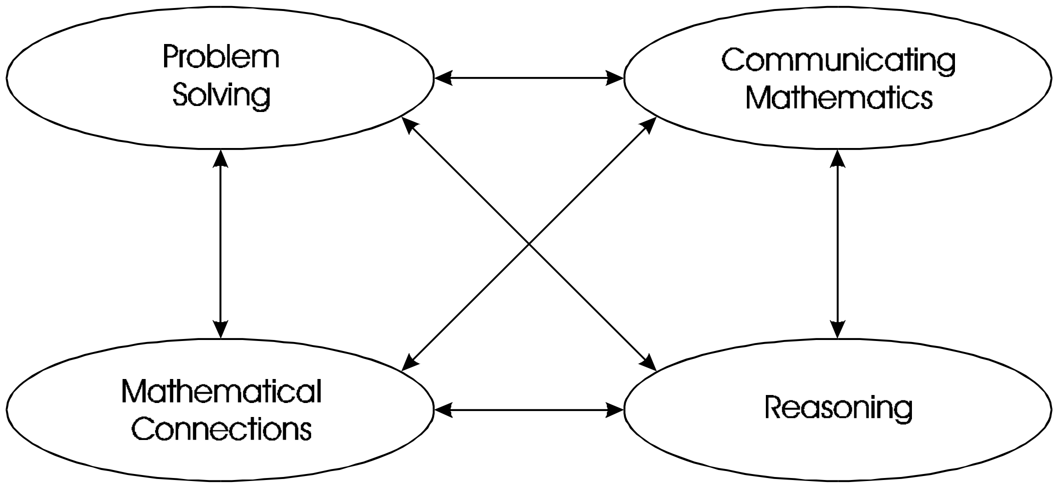
4. Justify answers and solution processes in a variety of problems.

- Students in both vignettes justify their answers and solution processes.

5. Analyze mathematical situations by recognizing and using patterns and relationships.

- The students in *Will a Dinosaur Fit?* solve their problems by comparing the sizes of the various dinosaurs with other sizes, such as the classroom and its doorway. The students in *Shapetown* recognize and use patterns and relationships as they pose and solve their problems involving attribute blocks.

The First Four Standards — Grades 3-4



Overview

In the third and fourth grades, students continue to develop their ability to solve problems, communicate mathematically, make connections within mathematics and between mathematics and other subject areas, and reason mathematically.

Students in grades 3-4 should continue to focus on understanding in their **problem solving** activities but should also begin to develop a repertoire of strategies for solving problems. These should include not only drawing a picture, using concrete objects, and writing a number sentence, but also drawing a diagram, working backwards, solving a simpler problem, and looking for a pattern. Students begin to spend more time developing a problem-solving plan, since they now have a greater variety of strategies to consider and select from. They also focus more on looking back, comparing each problem to ones they have solved previously.

Communication activities become more elaborate in third and fourth grade, as students become more comfortable with symbolic and written representations of ideas. Students should communicate with each other about mathematics on a daily basis, exploring problem situations and justifying their solutions. Different types of writing assignments may be used: keeping journals, explaining solutions to math problems, explaining mathematical ideas, and writing about the reasoning involved in solving a problem. Students continue to use manipulatives to explore new ideas and learn to relate different representations of an idea to each other. For example, after using base ten blocks to solve *7* x *36*, students might provide a pictorial representation of these blocks (at left below) followed by a written explanation of what they did to get *7* x *36 = 252*. Linking the use of concrete manipulatives to the pictorial and symbolic representations is critical to understanding the mathematical procedures.

***•••••• ***•••••• ***••••••	36 x 7	I laid out 7 groups of 3 tens and 6 ones.
***•••••• ***••••••	210	I counted up 7 x 3 = 21 tens and wrote down 210.
***•••••• ***••••••	42	I counted up 7 x 6 = 42 ones and wrote down 42.
	252	I added those together to get 252.

Children in third and fourth grade continue to build **mathematical connections.** Within mathematics, the major unifying ideas continue to be quantification (how much and how many, especially with larger quantities), patterns, and representing quantities and shapes. For example, students need to see the relationship between the quantification that they do with measurement (using centimeters and meters) and that they do with base ten blocks (representing numbers in the hundreds). Literature and social studies continue to provide opportunities for using mathematics in context. Students are also able to use mathematics more in their study of science, doing computations with the measurements they have made (e.g., averages). Measurement and data analysis, in particular, offer good opportunities for integrating science and mathematics. For example, students might measure the distance a hungry mealworm crawls in 90 seconds and compare it to the distance a well-fed mealworm crawls in the same amount of time.

Third- and fourth-graders use both **inductive reasoning** (looking for patterns, making educated guesses, forming generalizations) and **deductive reasoning** (using logical reasoning, eliminating possibilities, justifying answers). Teachers should create situations in which students may form incorrect generalizations based on only a few examples, and should be prepared to provide counter-examples to those incorrect generalizations. For example, if fourth-graders think that multiplying by 100 always means they add two zeros to the right side of the number, then the teacher should ask them to multiply 0.5 by 100 on their calculators. Instructional activities should continue to emphasize that mathematics makes sense and that mathematical reasoning helps people both to understand their world and to make decisions rationally.

Students in grades 3 and 4 continue to develop more formal and abstract notions of problem solving, communication, mathematical connections, and reasoning. They begin to focus more on what they are thinking as their communication and reasoning skills improve. They solve a wider range of problems and connect mathematics to a greater variety of situations in other subject areas and in life.

On-Line Resources

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The *Framework* will be available at this site during Spring 1997. In time, we hope to post additional resources relating to this standard, such as grade-specific activities submitted by New Jersey teachers, and to provide a forum to discuss the *Mathematics Standards*.

The First Four Standards — Grades 3-4

Vignette — Tiling a Floor

Standards: In addition to The First Four Standards, this vignette highlights Standards 7 (Geometry) and 10 (Estimation).

The problem: The third grade students toured the school and the playground to find and sketch the geometric shapes that they saw. On returning to the classroom, the class discussed names for each shape, compared the shapes, and talked about where each shape had been found. Several of the shapes had been copied from tiles on walls and floors. The teacher used the tiling idea to challenge students to decide which of the shapes could be used to tile a floor or a wall. (The use of shapes to form a tiling pattern is often referred to as "tessellation.")

The discussion: Questions such as these were examined by the teacher and students, to help clarify the task: *What do you know about tiling a floor? Can shapes go on top of each other? Can there be spaces? What are the names of the shapes that we found? How could we check each shape to see if we could tile with it? Do you think we all have to solve this problem the same way? What materials could we use to make the shapes? How many copies of each shape do you think we will need?*

Solving the problem: Students worked in pairs over a two-day period. Each pair selected three shapes to test for tiling. Before copying each shape, students wrote in their journals, naming each shape they selected, predicting whether each shape could or could not be used as a tile, and estimating how many would be needed to cover one sheet of paper. The names of some of the unfamiliar shapes were taken from a poster that was hanging in the classroom. Students selected a variety of materials for making copies of their shapes. Some selected plain paper and used rulers to draw copies of their shapes, others selected grid paper, still others selected square or triangular dot paper. Some pairs recognized their shapes in the container of pattern blocks and used them instead. Several pairs used a computer drawing program and were able to create many copies of their shapes quickly and easily. After making and cutting out 5 or more copies of each shape, they attempted to tile sheets of paper with the shapes. Successful and unsuccessful tilings were glued to construction paper. Students checked their previous predictions and continued their journal entries, reflecting on their predictions.

Summary: At the end of the two-day working period, the tilings were sorted into two groups: successful and unsuccessful. Discussion began with the successful tilings. Each tiling was labeled with the name of the shape. The teacher had students talk about the similarities and differences among the successful tilings. Students noticed that many of the successful tilings were made with four-sided shapes, that all the triangles led to successful tilings, and that there were many more shapes that were unsuccessful in tiling than were successful. Similar ideas were discussed for the unsuccessful tilings. Then students tried to verbalize why some shapes could be used as tiles and others could not. They were able to generalize that shapes that could be used for tilings were able to fit around a point without leaving spaces and without overlapping. To close the activity, students wrote in their journals about this generalization using their own words.

The First Four Standards — Grades 3-4

Vignette — Sharing Cookies

Standards: In addition to The First Four Standards, this vignette highlights Standards 6 (Number Sense) and 8 (Numerical Operations).

The problem: The fourth-grade teacher was ready to introduce students to experiences with fractions. This problem was posed as a way to gather information about the ideas that each student already had about fractions:

You have 8 cookies to share equally among 5 people. How much will each person get?

The discussion: Discussion began when the teacher posed the question *Why is this a problem?* With some prompting, students began to realize that there were not enough cookies to give each person 2 whole cookies, but if they gave each person just 1 cookie, there would be some left over. Students concluded that they would have to give each person 1 whole cookie and some part of another cookie. They had realized that finding that part was the "problem." The next major question for the students was *What would you like to use to solve the problem?* Students made many suggestions: *get cookies and cut them, use linking cubes, draw a picture of 8 circles, use paper circles.* The teacher provided construction paper circles, linking cubes, and cookies with plastic knives.

Solving the problem: Working in groups of 3 or 4, students were told that each group was to decide which materials to use to solve the problem, and that each group would explain its solution using pictures and numbers. Finally, they were told that they would be asked to share their solution with the whole class. Students worked in their groups, most choosing to use the real cookies, until they felt comfortable with their solutions. This was one solution: give each person 1 cookie, divide the rest of the cookies into halves, give each person one of these halves, divide the remaining half of a cookie into 5 equal pieces, and give each person one of those pieces. Another solution was: divide all the cookies into halves, give each person three halves, divide the remaining half into 5 equal pieces, and give each person one of those pieces. Students wrote number sentences describing the amount of each person's share, but most found that they were unable to simplify the number sentences to determine how much cookie each person gets.

Summary: The summary discussion centered on how much cookie each person got. The teacher found that students were able to determine the size of the smallest piece of cookie (1/10), but they were unable to determine how much one cookie, 1/2 of a cookie, and 1/10 of a cookie were altogether. The teacher extended the discussion so that the class was able to explore what made the problem difficult and how the problem could be changed to make it easier.

The First Four Standards — Grades 3-4

Indicators

The cumulative progress indicators for grade 4 for each of the First Four Standards appear in boldface type below the standard. Each indicator is followed by a brief discussion of how the preceding grade-level vignettes might address the indicator in the classroom in grades 3 and 4. The Introduction to this *Framework* contains three vignettes describing lessons for grades K-4 which also illustrate the indicators for the First Four Standards; these are entitled *Elevens Alive!*, *Product and Process*, and *Sharing a Snack*.

> Standard 1. All students will develop the ability to pose and solve mathematical problems in mathematics, other disciplines, and everyday experiences.

Building upon knowledge and skills gained in the preceding grades, experiences in grades 3-4 will be such that all students:

1. **Use discovery-oriented, inquiry-based, and problem-centered approaches to investigate and understand mathematical content appropriate to early elementary grades.**
    - In *Tiling a Floor*, students investigate tiling with different geometric shapes in the context of tiling a floor. In *Sharing Cookies*, students begin their study of fractions by considering a real-life problem.

2. **Recognize, formulate, and solve problems arising from mathematical situations and everyday experiences.**
    - In both vignettes, the students begin with a problem that arises from everyday experiences.

3. **Construct and use concrete, pictorial, symbolic, and graphical models to represent problem situations.**
    - In *Tiling a Floor*, the students use concrete materials (copies of shapes) to represent the problem situation. In *Sharing Cookies*, the students use a variety of manipulatives (paper circles and real cookies) to help them understand the problem.

4. **Pose, explore, and solve a variety of problems, including non-routine problems and open-ended problems with several solutions and/or solution strategies.**
    - In *Tiling a Floor*, the students use a variety of shapes in their exploration, although most use similar strategies (either concrete materials or a software program). The students in *Sharing Cookies* solve the same problem in a variety of ways. Although they recognize that the different solution methods lead to the same answer, they are unable to explain what fraction that is.

5. **Construct, explain, justify, and apply a variety of problem-solving strategies in both**

cooperative and independent learning environments.

- The students in the tiling vignette work in pairs, applying different strategies to investigate which shapes would tile the floor. The students in the cookie vignette work in groups of three or four; they construct a strategy for solving the problem, explain it, and justify it.

6. **Verify the correctness and reasonableness of results and interpret them in the context of the problems being solved.**

   - In both vignettes, the students share their results with the whole class. This sharing serves the purpose of verifying the correctness and reasonableness of these results.

7. **Know when to select and how to use grade-appropriate mathematical tools and methods (including manipulatives, calculators and computers, as well as mental math and paper-and-pencil techniques) as a natural and routine part of the problem solving process.**

   - Students in both vignettes are encouraged to select the tools they wish to use to solve the problems.

8. **Determine, collect, organize, and analyze data needed to solve problems.**

   - The students in *Tiling a Floor* decide what shapes they want to examine, collect data about which ones work and which ones do not, organize this information, and begin the analysis of the results.

9. **Recognize that there may be multiple ways to solve a problem.**

   - In *Tiling a Floor,* the students select a variety of materials, including a computer program, to solve the problem. In *Sharing Cookies,* the students all use different methods to solve the same problem.

Standard 2. All students will communicate mathematically through written, oral, symbolic, and visual forms of expression.

Building upon knowledge and skills gained in the preceding grades, experiences in grades 3-4 will be such that all students:

1. **Discuss, listen, represent, read, and write as vital activities in their learning and use of mathematics.**

   - The students in *Tiling a Floor* discuss the problem both initially and at the end, represent the problem situation using manipulatives or the computer, represent their solutions pictorially, and write about the solutions in their journals. The students in *Sharing Cookies* discuss the problem before and after working in groups, represent the problem situation and their solution using manipulatives or a picture and a number sentence, and write about their solution.

2. **Identify and explain key mathematical concepts, and model situations using oral, written,**

concrete, pictorial, and graphical methods.

- The students in the tiling lesson identify and explain the key concept of tiling from geometry. They use oral, written, concrete, and pictorial methods. In the cookie lesson, the students identify and explain the key concept of fractions (from number sense). They use oral, written, concrete, and pictorial methods.

3. **Represent and communicate mathematical ideas through the use of learning tools such as calculators, computers, and manipulatives.**

   - The students in *Tiling a Floor* use manipulatives and computers. The students in *Sharing Cookies* use manipulatives.

4. **Engage in mathematical brainstorming and discussions by asking questions, making conjectures, and suggesting strategies for solving problems.**

   - In both vignettes, the students brainstorm to clarify the task. They also make conjectures and suggest strategies for solving the problem.

5. **Explain their own mathematical work to others, and justify their reasoning and conclusions.**

   - In both vignettes, the students explain their solutions to the class and justify their work.

Standard 3. All students will connect mathematics to other learning by understanding the interrelationships of mathematical ideas and the roles that mathematics and mathematical modeling play in other disciplines and in life.

Building upon knowledge and skills gained in the preceding grades, experiences in grades 3-4 will be such that all students:

1. **View mathematics as an integrated whole rather than as a series of disconnected topics and rules.**

   - The students in *Tiling a Floor* tour the school and playground to find and discuss shapes, and then link these to discovering which shapes might be used for tiling a floor or wall. In *Sharing Cookies*, the students examine fractions as an application of a real-life problem rather than as an isolated topic in the book.

2. **Relate mathematical procedures to their underlying concepts.**

   - In *Tiling a Floor* students are able to discover that tiles which fit around a point without leaving spaces can be used as a pattern for tiling the floor. The students in *Sharing Cookies* will, in the near future, be learning how to simplify their answers by using addition to find a single fraction that describes their answer.

3. **Use models, calculators, and other mathematical tools to demonstrate the connections among various equivalent graphical, concrete, and verbal representations of mathematical concepts.**

    - Students in *Tiling a Floor* might use a software program such as *Tesselmania!* to create their own successful tessellations. In *Sharing Cookies*, the students use models to demonstrate that their answers are the same even though the number sentences are different.

4. **Explore problems and describe and confirm results using various representations.**

    - The students in both vignettes use shapes to explore their problems and describe their results in pictures and words.

5. **Use one mathematical idea to extend understanding of another.**

    - The tiling vignette uses the idea of shape to extend understanding of tiling. The cookie vignette uses division to build understanding of fractions.

6. **Recognize the connections between mathematics and other disciplines, and apply mathematical thinking and problem solving in those areas.**

    - The tiling vignette demonstrates the connection between mathematics and art. The cookie vignette connects mathematics to home economics where fractions are often used.

7. **Recognize the role of mathematics in their daily lives and in society.**

    - Both vignettes illustrate the use of mathematics in daily life.

Standard 4. All students will develop reasoning ability and will become self-reliant, independent mathematical thinkers.

Building upon knowledge and skills gained in the preceding grades, experiences in grades 3-4 will be such that all students:

1. **Make educated guesses and test them for correctness.**

    - The students in *Tiling a Floor* predict whether each shape can or can not be used as a tile. The students in *Sharing Cookies* might investigate how many different cookie amounts they can find which can be shared equally among five people (or 4 people, or 7, 8, and so on) with none left over. They guess that every multiple of the number of people would work, and use models for several examples to show that each number they select can be separated into piles containing equal amounts with none left over.

2. **Draw logical conclusions and make generalizations.**

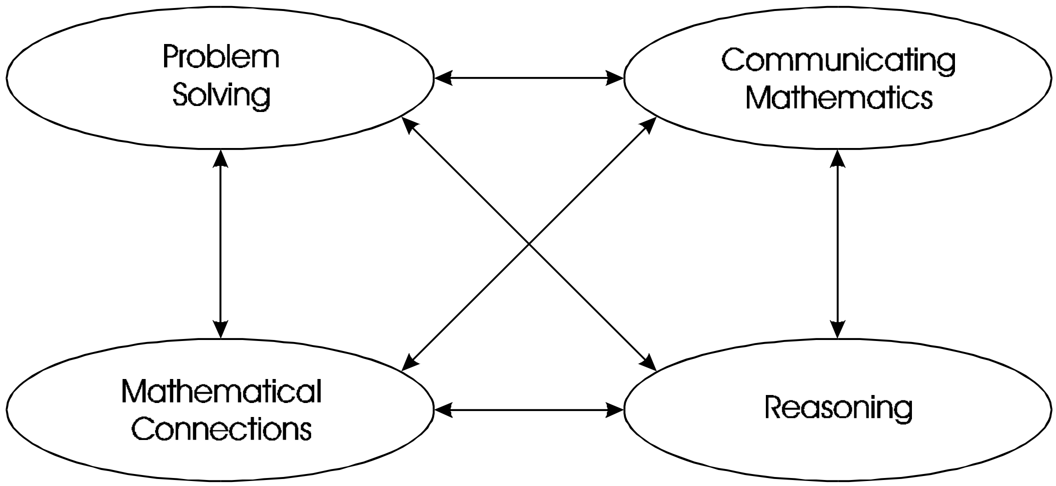
    - The students in the tiling vignette make a generalization that shapes which can be used for tilings fit around a point without leaving spaces or being on top of each other. Students in the cookie vignette conclude that 8 cookies are not enough to give each of the 5 people two whole cookies, but that each could have 1 cookie with some left over.

3. **Use models, known facts, properties, and relationships to explain their thinking.**
    - Students in both vignettes use models to explain their thinking.

4. **Justify answers and solution processes in a variety of problems.**
    - Students in both vignettes explain their answers and how they got them.

5. **Analyze mathematical situations by recognizing and using patterns and relationships**.
    - Students in the tiling vignette recognize that any triangle can be used as a tile. Students in the cookie vignette realize that each solution names the same fraction with a number sentence.

The First Four Standards — Grades 5-6



Overview

By grades 5-6, students have mastered the basics of problem solving, communication, mathematical connections, and reasoning, and can apply these with reasonable facility to familiar topics. They are ready to consider more complex tasks in each of these areas as well as to apply their skills to more advanced mathematical topics such as probability, statistics, geometry, and rational numbers.

Students in grades 5 and 6 should have experiences with a wide variety of **problem-solving** situations. Some of these should be process problems, in which the strategies students use are of more interest than the solution. For example: *The Boys' Club held its annual carnival last weekend. Admission to the carnival was $3 for adults and $2 for children under 12. Total attendance was 100 people and $232 was collected. How many adults and how many children attended the carnival?* (Kroll & Miller, 1993, p. 60)

Charles & Lester (1982) recommend that students first discuss a problem as a whole class, focusing on understanding the problem and discussing possible strategies to use in solving the problem. While students are working on the problem, the teacher should observe and question students, offering hints or problem extensions as needed. After the students have solved the problem, they should show and discuss their solutions and relate the problem to others previously solved. For example, the problem above is similar in structure to the one involving pigs and chickens that was discussed in the K-12 Overview.

Fifth- and sixth-graders should apply their improved **communication** skills to the new mathematics they are learning in order to help them better understand the concepts and procedures and to help their teachers better assess the students' understanding. Students should not only talk with each other in pairs, small groups, and as a class, they should also use mathematical symbols and language and draw pictures, diagrams, and other visual representations. Writing assignments may include keeping a journal or log, writing about mathematical concepts or procedures, or explaining how they solved a problem.

Students in the middle grades have a wider range of **mathematical connections** to address than do younger students. Within mathematics, multiple representations and patterns and generalizations are particularly important. For example, students might consider the numbers in the table below, looking for patterns and trying to develop a generalization (rule) that describes how the numbers are related.

x	**0**	**1**	**2**	**5**
y	**0**	**3**	**6**	**15**

They should consider not only the number sentence $y = 3x$ that describes this pattern but also the verbal rule *y is three times as big as x* and the graph, which involves a straight line. Such activities not only develop communication skills, they also address patterns, numerical operations, and algebra.

Students in grades 5 and 6 also apply mathematics to other subject areas and to real life. Social studies offers many opportunities for using data analysis and discrete mathematics, while art activities require the application of geometry. In science, students should observe and experiment with scientific phenomena and then summarize their observations using graphs, symbols, and geometry; they should translate the patterns they observe into mathematical terms. Following are several examples: *Students might compare the amount of bounce that results from dropping a ball from various heights, develop a rule relating the height of bounce to the height of the drop, and use the rule to predict how high the ball will bounce from any given height. They might repeat the activity for a variety of different balls. Students might learn about how numbers are used as rates by having them measure the distance a toy car goes down an inclined ramp and the amount of time it takes, and then compute the speed. The study of angles might be related to reflections of light in a mirror or the construction of sundials, students might use graphs and charts concerning relative humidity or wind chill, they might examine magnification as a real-world context for multiplication, they might connect volume to studying about water or density, or they might use pulse rates or the mean temperature or rainfall as examples of averaging.* Each of these activities demonstrates for students the usefulness of mathematics in learning about and doing science.

Students in fifth and sixth grade continue to develop their basic **reasoning** skills. Much emphasis at these grade levels should be placed on inductive reasoning in which students look for patterns, make and test conjectures, and form generalizations based on their observations. Their ability to use deductive reasoning — to use logic and justify conclusions — is enhanced at these grade levels by the teacher's frequent use of this type of reasoning. For example: *A square is a kind of rectangle. A rectangle is a kind of parallelogram. So a square is a kind of parallelogram.*

Students in the middle grades must be encouraged constantly to use their reasoning skills not only in mathematics class but also in other subjects and in their daily lives. Only in this way will they come to recognize the power of mathematical reasoning.

As fifth- and sixth-graders expand their understanding of mathematics to include more advanced topics, they also expand their understanding of problem solving, communication, mathematical connections, and reasoning. They apply these skills to more difficult problems concerning topics they have already studied and to problems involving new mathematical concepts and procedures. They further use these skills to gain a better understanding of new ideas.

References

Charles, R., and F. Lester. *Teaching Problem Solving: What, Why, & How.* Palo Alto, CA: Dale Seymour Publications, 1982.

Kroll, D. L., and T. Miller. “Insights from Research on Mathematical Problem Solving in the Middle Grades.” In *Research Ideas for the Classroom: Middle Grades Mathematics*. (D. T. Owens, Ed.) Reston, VA: National Council of Teachers of Mathematics, 1993.

On-Line Resources

http://dimacs.rutgers.edu/nj_math_coalition/framework.html/

The *Framework* will be available at this site during Spring 1997. In time, we hope to post additional resources relating to this standard, such as grade-specific activities submitted by New Jersey teachers, and to provide a forum to discuss the *Mathematics Standards*.

The First Four Standards — Grades 5-6

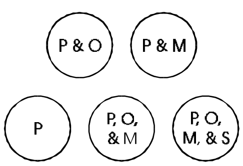
Vignette — Pizza Possibilities

Standards: In addition to The First Four Standards, this vignette highlights Standards 8 (Numerical Operations), 11 (Patterns), and Discrete Mathematics (14).

The problem: In a multi-aged group of fifth- and sixth-graders, the teacher posed this problem. *Pizza Shack has asked us to help design a form to keep track of certain pizza choices. They offer a cheese pizza with tomato sauce. A customer can then select from the following toppings: pepperoni, sausage, mushrooms, and onions. How many different choices for pizza does a customer have? List all the possible choices. Find a way to convince each other that you have accounted for all the possible choices.*

The discussion: To check for understanding, the teacher had various students restate the problem. Students began the discussion thinking that there were only 4 choices. One student suggested that there could be a pizza with both pepperoni and sausage. Students then realized the difficulty of the problem. There was a question about whether a pizza with no toppings was allowed. Students brainstormed about ways to solve the problem. Some suggestions were: draw pictures, make a list, make a chart, use cubes of different colors to represent the different toppings, and use letters or numbers to represent each topping. Before the groups began to solve the problem, the teacher had students predict and record the number of possible choices.

Solving the problem: Students worked in pairs for 15 to 20 minutes. Some groups had divided the work, with one student doing the 1- and 2-topping pizzas and the other the 3- and 4-topping pizzas. Others did the work separately and then compared their results. Still others had one person use cubes to show each combination as the other person recorded the combinations. Students kept track of the different kinds of pizzas in several ways.



P	M	S	O
0	0	0	0
1	0	0	0
0	1	0	0
0	0	1	0

pepperoni and sausage
pepperoni and mushrooms
pepperoni and onions
no toppings

Summary: Pairs of students reported to the class about how they solved the problem, and they compared their predictions with their actual results. They had some success giving convincing arguments, mostly relying on well-organized listing procedures. One pair of students suggested that they knew their solution was correct because it was part of the following pattern which involves powers of 2: if no toppings are available, there is only one pizza; if one topping is available, there are two possible pizzas; if two toppings are available, there are four possible pizzas; if three toppings are available, there are eight possible pizzas; and if four toppings are available, there are sixteen possible pizzas. With each extra topping, they correctly reasoned, the number of possibilities doubled, since each pizza on the previous list could now also be made with the additional topping. They confirmed their reasoning by constructing a chart like that above with 16 different rows, each row representing one possible pizza.

The First Four Standards — Grades 5-6

Vignette — Two-Toned Towers

Standards: In addition to the First Four Standards, this vignette highlights Standards 7 (Geometry), 8 (Numerical Operations), 11 (Patterns), 13 (Probability and Statistics), and 14 (Discrete Mathematics).

The problem: The same multi-aged group which worked on the pizza problem was challenged with this towers problem several weeks later. *Your group's task is to build as many towers as you can that are 4 cubes tall and that use no more than 2 colors. Then you are to convince each other that there are no duplicates and none has been omitted.*

The discussion: Students questioned the meaning of "no more than 2 colors." Some thought that each tower must have 2 colors, other thought that towers of 1 color were also allowed. After some discussion, students agreed, for this problem, to build towers with 1 or 2 colors. The teacher then asked two volunteers to build two towers each — one that satisfied the conditions of the problem and the other that did not. Students discussed each tower, and decided which ones were appropriate. Some students wondered if they had to build the towers, or if they could just draw and label the towers. The group decided that not actually building the towers was okay, as long as each student kept a permanent record of the towers that he "built."

Solving the problem: Students worked in pairs or other small groups to begin building the towers. In most groups, at least one person felt more comfortable actually building the towers, rather than relying on drawing the towers. As the groups worked, the teacher stopped at each group, and encouraged students to talk about how they knew when all the towers were made and how they knew there were no duplicates. Some students working with red and blue cubes were very organized, building towers in this order: RRRR, BRRR, RBRR, RRBR, RRRB, and so on. Others worked very haphazardly, and did not organize their towers until they were all made. Still others used an "opposites" approach, making the RRRB tower followed by BBBR, RBBR followed by BRRB, and so on.

Summary: Students used their solutions to the problem as a focal point for their reports to the class. All of the students agreed that there were 16 towers and were able to convince their classmates by using some variation of the organized list strategy. Then, one student suggested that somehow the pizza problem and the towers problem were alike. Students grappled with the idea for a few minutes. Then the teacher suggested that they list the things that were alike about the problems. Their list included these ideas: there were 4 toppings in the pizza problem and 4 blocks in each tower; 0 toppings was like a tower of all one color and 4 toppings was like a tower of all the other color. After more discussion, students realized that they could better match up the two problems if they put the toppings in a specific order, like pepperoni, mushrooms, sausage, and onions. Then if you ordered a pizza with only pepperoni and mushrooms, it would be like saying "yes, yes, no, no" to the four toppings; and that was like building a tower which was "red, red, blue, blue." A pizza with pepperoni and sausage, or "yes, no, yes, no", would match up to the tower of "red, blue, red, blue." As the discussion went on, students went back to their problem-solving groups to record the connections between the two problems. At the end of the class, the teacher explained that both of these problems belong to an area of mathematics called *combinatorics*, which deals with problems involving combinations and is used in analyzing games of chance.

The First Four Standards — Grades 5-6

Indicators

The cumulative progress indicators for grade 8 for each of the First Four Standards appear in boldface type below the standard. Each indicator is followed by a brief discussion of how the preceding grade-level vignettes might address the indicator in the classroom in grades 5 and 6. The Introduction to this *Framework* contains three vignettes describing lessons for grades 5-8 which also illustrate the indicators for the First Four Standards; these are entitled *The Powers of the Knight*, *Short-circuiting Trenton*, and *Mathematics at Work*.

> Standard 1. All students will develop the ability to pose and solve mathematical problems in mathematics, other disciplines, and everyday experiences.

Building upon knowledge and skills gained in the preceding grades, experiences in grades 5-6 will be such that all students:

4[*]. **Pose, explore, and solve a variety of problems, including non-routine problems and open-ended problems with several solutions and/or solution strategies**.

- Both vignettes involve problems with multiple solution strategies.

5[*]. **Construct, explain, justify, and apply a variety of problem-solving strategies in both cooperative and independent learning environments.**

- In both vignettes, students apply, explain, and justify their choice of problem-solving strategies. In each case, the students work in pairs or small groups.

6[*]. **Verify the correctness and reasonableness of results and interpret them in the context of the problems being solved.**

- Students in both vignettes verify their results by sharing them with each other. Since the problems involve single answers, the students decide that they are correct if they all agree and can understand each other's reasoning.

7[*]. **Know when to select and how to use grade-appropriate mathematical tools and methods (including manipulatives, calculators and computers, as well as mental math and paper-and-pencil techniques) as a natural and routine part of the problem-solving process.**

- Students in both vignettes use manipulatives as a part of the problem-solving process.

[*] Reference is made here to Indicators 4, 5, 6, 7, and 8, which are also listed for grade 4, since the Standards specify that students demonstrate continued progress in these indicators.

8[*]. Determine, collect, organize, and analyze data needed to solve problems.

- The students in *Pizza Possibilities* determine the different possibilities, organize this information in a table, and analyze their results. The students in *Two-Toned Towers* make organized lists and analyze their results.

10. Use discovery-oriented, inquiry-based, and problem-centered approaches to investigate and understand mathematical content appropriate to the middle grades.

- Both vignettes deal with the same content, determining the number of possible combinations (discrete mathematics). The topic is introduced through two different problem situations which are then compared to each other.

11. Recognize, formulate, and solve problems arising from mathematical situations, everyday experiences, and applications to other disciplines.

- The pizza problem arises from the students' everyday experiences, while the tower problem arises from a more fanciful situation.

12. Construct and use concrete, pictorial, symbolic, and graphical models to represent problem situations and effectively apply processes of mathematical modeling in mathematics and other areas.

- The students in both vignettes use concrete materials, pictures, lists, or tables to represent the problem situation.

13. Recognize that there may be multiple ways to solve a problem, weigh their relative merits, and select and use appropriate problem-solving strategies.

- Students in both vignettes share their different strategies for solving the problem.

14. Persevere in developing alternative problem-solving strategies if initially selected approaches do not work.

- In the pizza vignette, students begin the discussion thinking that there are only 4 choices, but after futher discussion and brainstorming they are able to use various ways to solve the problem. In the towers vignette, some students are not comfortable with drawing pictures and so use concrete materials instead.

Standard 2. All students will communicate mathematically through written, oral, symbolic, and visual forms of expression.

Building upon knowledge and skills gained in the preceding grades, experiences in grades 5-6 will be such that all students:

[*] Reference is made here to Indicators 4, 5, 6, 7, and 8, which are also listed for grade 4, since the Standards specify that students demonstrate continued progress in these indicators.

1[*]. **Discuss, listen, represent, read, and write as vital activities in their learning and use of mathematics.**

- Students in *Pizza Possibilities* discuss the problem before and after solving it, listen to each other, represent their solutions in different ways, and write about their solutions. Students in *Two-Toned Towers* also discuss, listen, represent, and write about their solutions.

2[*]. **Identify and explain key mathematical concepts, and model situations using oral, written, concrete, pictorial, and graphical methods.**

- Students in both vignettes use oral, written, concrete, and pictorial methods to explain the mathematical ideas involved in their problems. The oral discussion in the towers vignette might be followed up by asking students to write about what they learned concerning combinations.

3[*]. **Represent and communicate mathematical ideas through the use of learning tools such as calculators, computers, and manipulatives.**

- Both vignettes involve the use of manipulatives.

4[*]. **Engage in mathematical brainstorming and discussions by asking questions, making conjectures, and suggesting strategies for solving problems.**

- In both vignettes, the teacher involves the class in discussion before solving the problem.

5[*]. **Explain their own mathematical work to others, and justify their reasoning and conclusions.**

- In both vignettes, students explain their work to their classmates, justifying their solutions.

6. **Identify and explain key mathematical concepts and model situations using geometric and algebraic methods.**

- In both vignettes, students use cubes of different colors to model the number of possibilities geometrically.

7. **Use mathematical language and symbols to represent problem situations, and recognize the economy and power of mathematical symbolism and its role in the development of mathematics.**

- Students use symbols to represent the toppings in the pizza vignette and the colors of the cubes in the towers vignette. They also make the connections between these symbols in discussing how the two problems were alike.

8. **Analyze, evaluate, and explain mathematical arguments and conclusions presented by others.**

- Students in both vignettes analyze each other's arguments and evaluate them.

[*] Reference is made here to Indicators 1, 2, 3, 4, and 5, which are also listed for grade 4, since the Standards specify that students demonstrate continued progress in these indicators.

Standard 3. All students will connect mathematics to other learning by understanding the interrelationships of mathematical ideas and the roles that mathematics and mathematical modeling play in other disciplines and in life.

Building upon knowledge and skills gained in the preceding grades, experiences in grades 5-6 will be such that all students:

1[*]. View mathematics as an integrated whole rather than as a series of disconnected topics and rules.

- By drawing students' attention to the connection between two seemingly unrelated problems, the teacher encourages the students to look for connections between mathematical topics.

2[*]. Relate mathematical procedures to their underlying concepts.

- In both vignettes, the students relate the problem to making lists of possibilities and to taking powers of 2.

3[*]. Use models, calculators, and other mathematical tools to demonstrate the connections among various equivalent graphical, concrete, and verbal representations of mathematical concepts.

- In both vignettes, students use models to represent the situation. After describing and representing the problems concretely, they discover that their models are equivalent and that the problems are essentially the same.

4[*]. Explore problems and describe and confirm results using various representations.

- In both vignettes, students use a variety of representations to explore the problem situation and to describe their results.

8. Recognize and apply unifying concepts and processes which are woven throughout mathematics.

- Both vignettes deal with the unifying concepts of patterns and the process of making an organized list.

9. Use the process of mathematical modeling in mathematics and other disciplines, and demonstrate understanding of its methodology, strengths, and limitations.

- The students develop a very simple mathematical model when they introduce symbols to represent the ingredients on the pizza and record the different combinations. Once they notice the pattern of powers of two the students develop a numerical representation of their model. In *Two-Toned Towers*, students use essentially the same simple model to create the different towers. Then they link these representations to the pizza topping problem and also to "yes, no" choices for selecting items.

[*] Reference is made here to Indicators 1, 2, 3, and 4, which are also listed for grade 4, since the Standards specify that students demonstrate continued progress in these indicators.

10. Apply mathematics in their daily lives and in career-based contexts.

- The students in both vignettes apply mathematics to a realistic problem.

11. Recognize situations in other disciplines in which mathematical models may be applicable, and apply appropriate models, mathematical reasoning, and problem solving to those situations.

- Students might investigate in which other disciplines the concept of continual doubling occurs. For example, in science, bacteria cells divide, and thus double, repeatedly.

Standard 4. All students will develop reasoning ability and will become self-reliant, independent mathematical thinkers.

Building upon knowledge and skills gained in the preceding grades, experiences in grades 5-6 will be such that all students:

2*. Draw logical conclusions and make generalizations.

- At the end of the towers vignette, as the students compare the two problems, they begin to draw logical conclusions about the relationship between the problems. The teacher might further extend these generalizations by building upon the numerical pattern observed by students in the pizza problem. For example, a tower one cube high can be built in two ways with two colors, a tower two cubes high can be built in 2 x 2 = 4 ways, one three cubes high can be built in 2 x 2 x 2 = 8 ways, and one four cubes high can be built in 2 x 2 x 2 x 2 = 16 ways. *In how many ways could you build a tower ten cubes high with two colors? In how many ways could you build a tower three cubes high with five colors?*

3*. Use models, known facts, properties, and relationships to explain their thinking.

- In both vignettes, the students use their models to explain their thinking.

5*. Analyze mathematical situations by recognizing and using patterns and relationships.

- In both vignettes, the students look for patterns to help them solve the problem.

6. Make conjectures based on observation and information, and test mathematical conjectures and arguments.

- In both vignettes, the students make conjectures about the problem solution based on observation and test their conjectures.

7. Justify, in clear and organized form, answers and solution processes in a variety of problems.

- Students in both vignettes organize their results and justify their solutions.

* Reference is made here to Indicators 2, 3, and 5, which are also listed for grade 4, since the Standards specify that students demonstrate continued progress in these indicators.

8. **Follow and construct logical arguments, and judge their validity.**

   - Students construct logical arguments in explaining their solutions and understand and agree with the arguments presented by other students in both vignettes.

9. **Recognize and use deductive and inductive reasoning in all areas of mathematics.**

   - The students use both types of reasoning. First, they use inductive reasoning as they look for patterns in the problems. They begin to use deductive reasoning in justifying their results and further extend their use of inductive and deductive reasoning as they discuss how the two problems are related.

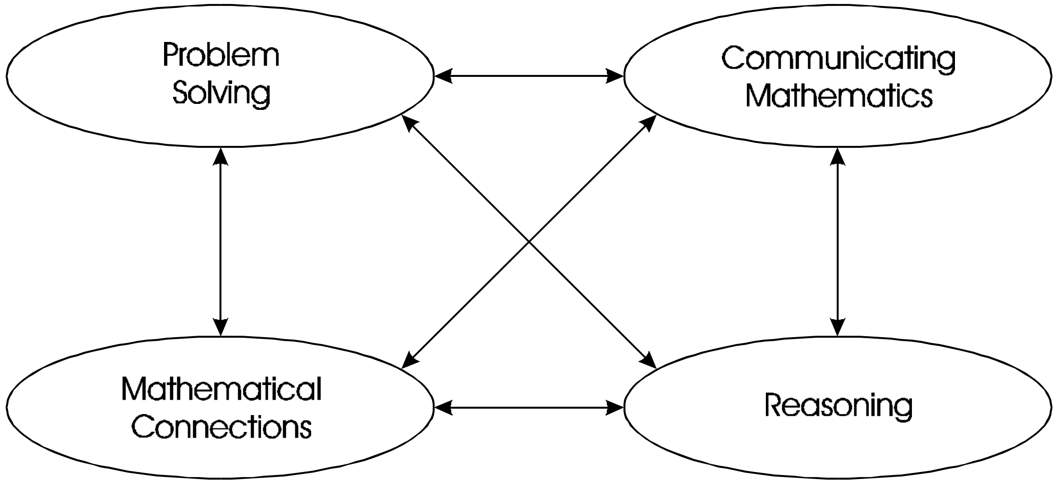
10. **Utilize mathematical reasoning skills in other disciplines and in their lives.**

    - The pizza and related towers vignette demonstrate the use of mathematical reasoning in real life.

11. **Use reasoning rather than relying on an answer-key to check the correctness of solutions to problems.**

    - The students in both vignettes decide whether or not their answers are correct by comparing them and listening to each other's justifications.

The First Four Standards — Grades 7-8



Overview

Seventh- and eighth-graders continue to improve their mastery of problem solving, communication, mathematical connections, and reasoning. They are increasingly able to apply these skills in more formal and abstract situations.

Students in grades 7-8 need to have many opportunities to practice **problem solving**, reflecting upon their thinking and explaining their solutions. They should have developed a considerable repertoire of problem-solving strategies by this time, including working backwards, writing an equation, looking for patterns, making a diagram, solving a simpler problem, and using concrete objects to represent the problem situation. Students should be able to apply the steps of understanding the problem, making a plan, carrying out the plan, and looking back. (See the K-12 Overview at the beginning of this chapter for a discussion of these steps.)

Students in grades 7 and 8 need many opportunities to **communicate** mathematically. Explaining and justifying one's work orally and in writing leads to deeper understanding of concepts and principles. Through discussion, students reach agreement on the meanings of words and recognize the importance of having common definitions. The need for mathematical symbols arises from the exploration of concepts, and these concepts must be firmly connected to the symbols that represent them through frequent and explicit discussion of the relationships between concepts and symbols. Students must also be encouraged to construct connections among concepts, procedures, and approaches by using questions that require more elaborate communication skills. For example, students might give examples of: a rectangle with four congruent sides, a parallelogram with four right angles, a trapezoid with two equal angles, a number between 1/3 and 1/2, a number with a repeating decimal representation, a net for a cube, or an equation for a line that passes through the point (–1,2). (See *Curriculum and Evaluation Standards for School Mathematics*, p. 80.) [A net is a flat shape which when folded along indicated lines will produce a three-dimensional object; for example, six identical squares joined in the shape of a cross can be folded to form a cube. Tabs added to the net facilitate attaching appropriate edges so that the shape remains three-dimensional.]

Many students in the middle grades have in the past viewed mathematics as a collection of isolated skills to be memorized and later forgotten. These students must broaden their perspective, viewing mathematics as an integrated whole and acknowledging its relevance in and out of school. They must improve their understanding of **mathematical connections.** Students must understand, for example, that translations on the number line are fundamentally the same as adding numbers. They should connect the various interpretations of fractions to measurement, ratios, and algebra. They should link Pascal's triangle with counting, exponents, number patterns, algebra, geometric patterns, probability, and number theory. (See Standard 14, Discrete Mathematics.) Such connections can be enhanced by using technology and by exploring the same mathematical ideas in varied contexts. Some of the most important connections for students in grades 7 and 8 include proportional relationships (see Standard 6, Number Sense) and the relationship between data in tables and their algebraic and graphical representations. (See Standard 11, Patterns; Standard 12, Probability and Statistics; and Standard 13, Algebra).

Students in these grades also need many opportunities to discuss the connections between mathematics and other disciplines and the real world. For example, students concerned about traffic at a nearby intersection might design a study to collect data about the situation. Their study might involve first deciding what "traffic" is, how to count it, how to record the data, what the data means, how to remedy the situation, who is responsible for dealing with this type of situation, and how to convince that person that change is needed. Their study might end with a letter to the town council, recommending changes needed at the intersection.

In conjunction with studying about maps in social studies, for example, students might study scaling and its relationship to similarity, ratio, and proportion. Measurement situations arise in social studies, science, home economics, industrial technology, and physical education. Weather forecasting, scientific experiments, advertising claims, chance events, and economic trends offer more opportunities to relate mathematics to the real world, often through the use of statistics and probability. Connections between mathematics and science are plentiful:

- Students use symmetry and perspective to create original artistic work.
- Students might learn about scientific notation for large numbers by studying the solar system or for small numbers by studying viruses and bacteria.
- Different scientific contexts, such as electrical charges or forces in opposite directions, might help students develop multiple representations for integers.
- Students might plot earthquake locations by using a compass and protractor.
- The study of geometric shapes might be related to the study of crystals.
- Area and volume might be considered in conjunction with looking at the number of fish in a pond. *Why would each measure be important to consider? How could the area of a pond be found? How could the volume be found?*
- Students might investigate levers, gears, or pulleys, looking for numerical patterns and generating equations that describe the relationships found.

Students need frequent opportunities to explore and discuss many of these types of relationships in order to develop an appreciation for the power of mathematics.

Students in grades 7 and 8 experience rapid improvements in their ability to use mathematical **reasoning.** Special attention should be paid at these grades to proportional and spatial reasoning and reasoning from graphs. Students need to experience both inductive reasoning (making conjectures, predicting outcomes, looking for patterns, and making generalizations) and deductive reasoning (using logical arguments, justifying answers). Experiences with inductive reasoning need to include situations in which incorrect generalizations based on too few examples are tested. Instructional activities that address two specific types of deductive reasoning are appropriate at these grades. *Class reasoning* involves applying generalizations to a specific situation: for example, all even numbers are divisible by *2*; x is an even number; so x is divisible by *2*. *Conditional reasoning* involves using if-then statements. These types of reasoning need to be used on a regular basis in class discussions, assignments, and tests in order to help students become familiar with valid reasoning patterns. Students should also be asked to justify and explain solutions to the satisfaction of their peers.

In grades 7 and 8, students apply their problem-solving, communication, and reasoning skills in an increasingly diverse set of situations as they develop better understanding of the connections within mathematics and between mathematics and other subjects and the real world. In doing so, they learn new mathematical concepts and apply familiar mathematics to new situations.

References

National Council of Teachers of Mathematics. *Curriculum and Evaluation Standards for School Mathematics.* Reston, VA, 1989.

On-Line Resources

http://dimacs.rutgers.edu/nj_math_coalition/framework.html/

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The First Four Standards — Grades 7-8

Vignette — Sketching Similarities

Standards: In addition to the First Four Standards, this vignette highlights Standards 5 (Tools and Technology), 7 (Geometry), 9 (Measurement), and 15 (Building Blocks of Calculus).

The problem: A heterogeneously grouped class of middle school students reported to the computer lab for their mathematics class. Students had used *Geometer's Sketchpad* software (Key Curriculum Press) frequently during the year. They had been studying proportions in everyday and geometric situations during several previous class periods. They had also defined similar figures as those generated by an expansion or a contraction. The teacher presented this task to the class: *Draw several similar figures. What do you notice about the ratios of the measures of corresponding sides? What do you notice about the measures of the corresponding angles? Record your results and write your conclusions.*

The discussion: The teacher asked students to begin brainstorming to generate ideas about the problem. As ideas were suggested, the teacher wrote them on the chalkboard. Ideas included: *Use simple figures to make the problem easier. Use the dilate transformation. Use measure and length on Sketchpad to find the lengths of the sides. Make more than two figures because the problem says several. What is corresponding? Do we have to write complete sentences? Use measure and angle on Sketchpad to find the measures of the angles.* After all the ideas were recorded, the class discussed each one. Some were questions that could be answered by students, by rereading the problem or by checking glossaries in math books. Those answers were recorded next to the appropriate questions. Some of the ideas were eliminated because they were either incorrect or off-task. The teacher then had students restate the problem and, using the list, talk about the steps they would follow to solve the problem.

Solving the problem: Students joined their regular lab partners at the computers and began the investigation. As students worked, some requested teacher help, asking if their work was "finished yet" and if their results were "good enough." The teacher directed those students to reread the problem and the list of ideas developed during the discussion, and determine for themselves which parts of the problem were completed adequately and which were left to be done. Students worked for two class periods, completing the task by preparing diagrams, ratios, and written explanations. The teacher provided each pair with a transparency on which they were to record their diagrams and ratios to share with the class.

Summary: The summary discussion began with a volunteer restating the problem. Another volunteer described the steps she used to solve the problem. Others interjected during the description to tell how their methods differed. Then students presented their findings, using their transparencies and their written conclusions. After lengthy discussion, students wrote a general class conclusion that similar figures have corresponding angles with equal measures and have corresponding sides whose measures form equal ratios.

The First Four Standards — Grades 7-8

Vignette — Rod Dogs

Standards: In addition to the First Four Standards, this vignette highlights Standards 7 (Geometry), 8 (Numerical Operations), 9 (Measurement), 11 (Patterns), and 15 (Building Blocks of Calculus).

The problem: Students in this multi-aged middle school class had been working for several days on a project that gave them opportunities to find the surface area and volume of objects constructed with Cuisenaire rods and tacky putty. Today they were to begin an investigation of the relationship between the surface area and the volume of 3-dimensional objects enlarged with scale factors from 2 to 8. Students, working in groups of two or three, were instructed to build a "dog" using 1 yellow (5cm in length) and 5 red (2cm) rods. The yellow rod represented the dog's body, and a red rod represented each leg and the head. Then students calculated the surface area and volume of the dog. (Note that all rods have cross-section 1cm x 1cm.) Students were then instructed to build a "double dog" that was twice as big in each of the three dimensions. After several false starts, students constructed a dog with 4 orange (10cm) rods and 20 purple (4cm) rods. The 4 orange rods represented the dog's body, and 4 purple rods represented each leg and the head. Then students found and recorded the surface area and volume of this double dog. Each group was to build one other dog, using an assigned scale factor of 3, 4, 5, 6, 7, or 8 times the original dog. Their challenge was to determine how the scale factors were related to the surface area and volume of the enlarged dogs.

The discussion: Students asked questions to clarify the problem, mostly checking the task by saying *You mean, if my scale factor is 7, my dog's body has to be 35 by 7 by 7?* The teacher encouraged other students to confirm or correct each question. Then students focused on how they were to construct the dogs. Many began to think about how many rods they would need to complete the construction and decided that for some of the scale factors, they just would not have enough rods. The teacher then pointed out the centimeter grid paper, scissors, and tape at the front of the room and suggested that the students could build the dogs with those supplies.

Solving the problem: Students worked for two days building their enlarged dogs. Part of the challenge was to lay out a pattern for each part of the dog on a sheet of paper so that it could be folded and then put together with as few taped edges as possible. Those students who were given scale factors of 3 or 4 found the task rather simple and were able to count the surface area and volume without much trouble. Those with greater scale factors such as 7 or 8 had a more difficult task, but were able to complete it successfully.

Summary: The scale factor, surface area, and volume of each dog was listed on the chalkboard. Students discussed the various methods that different groups used to construct the dogs and to find the surface area and volume. The teacher challenged students to find the pattern in the chart, to apply the rule to scale factors other than those already used, and to generalize the rule to a scale factor of *n*. Students made liberal use of calculators, using a guess and check strategy to find the pattern. Students were able to verbalize that the surface area was the scale factor squared times the original surface area and that the volume was the scale factor cubed times the original volume.

The First Four Standards — Grades 7-8

Indicators

The cumulative progress indicators for grade 8 for each of the First Four Standards appear in boldface type below the standard. Each indicator is followed by a brief discussion of how the preceding grade level vignettes might address the indicator in the classroom in grades 7 and 8. The Introduction to this *Framework* contains three vignettes describing lessons for grades 5-8 which also illustrate the indicators for the First Four Standards; these are entitled *The Powers of the Knight*, *Short-circuiting Trenton*, and *Mathematics at Work*.

Standard 1. All students will develop the ability to pose and solve mathematical problems in mathematics, other disciplines, and everyday experiences.

Building upon knowledge and skills gained in the preceding grades, experiences in grades 7-8 will be such that all students:

4*. Pose, explore, and solve a variety of problems, including non-routine problems and open-ended problems with several solutions and/or solution strategies.

- The students in *Sketching Similarities* explore a problem involving geometry in which they choose their own figures to investigate. The students in *Rod Dogs* explore and solve a non-routine problem.

5*. Construct, explain, justify, and apply a variety of problem-solving strategies in both cooperative and independent learning environments.

- In both vignettes, students use the strategy of looking for patterns. The students in *Sketching Similarities* work in pairs and use inductive reasoning to find a pattern. The groups of students in *Rod Dogs* use concrete materials as their primary strategy, with some using rods and others using grid paper.

6*. Verify the correctness and reasonableness of results and interpret them in the context of the problems being solved.

- In both vignettes, the students verify their results and interpret them through a class discussion following their work in pairs or groups.

7*. Know when to select and how to use grade-appropriate mathematical tools and methods (including manipulatives, calculators and computers, as well as mental math and paper-and-pencil techniques) as a natural and routine part of the problem-solving process.

* Reference is made here to Indicator 4, 5, 6, 7, and 8, which are also listed for grade 4, since the Standards specify that students demonstrate continued progress in these indicators.

- Students in *Sketching Similarities* use computers. Students in *Rod Dogs* use manipulatives and calculators as needed in solving their problems.

8[*]. Determine, collect, organize, and analyze data needed to solve problems.

- The students in *Sketching Similarities* decide what data to collect in the initial discussion of the problem. They analyze the collected data in the discussion following their work in pairs. The students in *Rod Dogs* collect and organize the data as they work in groups. In their summary discussion, they analyze all the data collected to look for patterns and form a generalization.

10. Use discovery-oriented, inquiry-based, and problem-centered approaches to investigate and understand mathematical content appropriate to the middle ages.

- The students in both vignettes use a problem to investigate new mathematical concepts.

11. Recognize, formulate, and solve problems arising from mathematical situations, everyday experiences, — and applications to other disciplines.

- The *Sketching Similarities* vignette involves examining a problem that arises from a mathematical situation. The *Rod Dogs* vignette involves a problem that arises from everyday experiences — enlarging a three-dimensional object.

12. Construct and use concrete, pictorial, symbolic, and graphical models to represent problem situations and effectively apply processes of mathematical modeling in mathematics and other areas.

- The students in *Sketching Similarities* use graphical (computer diagrams) and symbolic (numbers) models in considering their problem. The students in *Rod Dogs* use concrete materials (rods and grid paper folded to make boxes) and symbolic (numbers and formulas) models.

13. Recognize that there may be multiple ways to solve a problem, weigh their relative merits, and select and use appropriate problem-solving strategies.

- The students in *Sketching Similarities* use brainstorming to generate ideas about solving the problem, and share their methods with the class. In *Rod Dog*s, students approach the problem in different ways, and the class discusses the various methods that different groups use to construct the "dogs" and to solve the problem; some students use a guess and check strategy to find the pattern.

14. Persevere in developing alternative problem-solving strategies if initially selected approaches do not work.

- When some students are at a standstill in *Rod Dogs,* the teacher directs their attention to available manipulatives. In *Sketching Similarities,* the teacher encourages students to research the problem and to make use of the list of ideas generated by the class in their

[*] Reference is made here tp Indicators 4, 5, 6, 7, and 8, which are also listed for grade 4, since the Standards specify that students demonstrate continued progress in these indicators.

continued attempts to solve the problem.

> Standard 2. All students will communicate mathematically through written, oral, symbolic, and visual forms of expression.

Building upon knowledge and skills gained in the preceding grades, experiences in grades 7-8 will be such that all students:

1[*]. Discuss, listen, represent, read, and write as vital activities in their learning and use of mathematics.

- Students in both vignettes discuss, listen, represent the problem situation concretely and symbolically, and read and write about their solutions.

2[*]. Identify and explain key mathematical concepts, and model situations using oral, written, concrete, pictorial, and graphical methods.

- Students in *Sketching Similarities* explain the concept of similarity and model the situation using oral, written, concrete, and pictorial methods. Students in *Rod Dogs* explain how the scale factor is related to changes in area and volume and use oral, written, concrete, and pictorial methods.

3[*]. Represent and communicate mathematical ideas through the use of learning tools such as calculators, computers, and manipulatives.

- Students in *Sketching Similarities* use computers and manipulatives. Students in *Rod Dogs* use manipulatives and calculators.

4[*]. Engage in mathematical brainstorming and discussions by asking questions, making conjectures, and suggesting strategies for solving problems.

- Students in both vignettes begin their activity by asking questions, making conjectures, and suggesting strategies to be used.

5[*]. Explain their own mathematical work to others, and justify their reasoning and conclusions.

- Students in both vignettes share their work with the class and justify their solutions.

6. Identify and explain key mathematical concepts and model situations using geometric and algebraic methods.

- In *Sketching Similarities* students recognize that similarity of triangles has an algebraic counterpart, that corresponding sides have the same ratio; the activities for the vignette could be extended by asking the students to use algebra to determine the lengths of the sides. In *Rod Dogs* the students develop a formula relating the volume and surface area of the

[*] Reference is made here to Indicators 1, 2, 3, 4, and 5, which are also listed for grade 4, since the Standards specify that students demonstrate continued progress in these indicators.

enlarged "dogs" to the volume and surface area of the original one.

7. **Use mathematical language and symbols to represent problem situations, and recognize the economy and power of mathematical symbolism and its role in the development of mathematics.**
   - Students in both vignettes use mathematical language and symbols to represent their solutions.

8. **Analyze, evaluate, and explain mathematical arguments and conclusions presented by others.**
   - Students in both vignettes analyze, evaluate, and explain their solutions.

> Standard 3. All students will connect mathematics to other learning by understanding the interrelationships of mathematical ideas and the roles that mathematics and mathematical modeling play in other disciplines and in life.

Building upon knowledge and skills gained in the preceding grades, experiences in grades 7-8 will be such that all students:

1*. **View mathematics as an integrated whole rather than as a series of disconnected topics and rules.**
   - The *Sketching Similarities* vignette illustrates the connection between similarity and ratios. The *Rod Dogs* vignette illustrates the connection between geometry and measurement (surface area, volume, scale factor) and exponents.

2*. **Relate mathematical procedures to their underlying concepts.**
   - The *Sketching Similarities* vignette relates the mathematical procedure of setting up ratios for similar figures to the underlying concept of similarity. The *Rod Dogs* vignette relates the mathematical procedure of taking powers of a number to the underlying concepts of area and volume.

3*. **Use models, calculators, and other mathematical tools to demonstrate the connections among various equivalent graphical, concrete, and verbal representations of mathematical concepts.**
   - Students in both vignettes use tools (models, computers, calculators) to discover, demonstrate, and verbalize the connections between the concrete geometric models in each problem and specific numerical relationships which result. In *Sketching Similarities*, they might also apply their conclusions to express and solve numerical and algebraic problems associated with the angles and sides of similar triangles. In *Rod Dogs*, students might use their discoveries regarding the 3-dimensional objects and graph the relationships of scale

* Reference is made here to Indicators 1, 2, 3, and 4, which are also listed for grade 4, since the Standards specify that students demonstrate continued progress in these indicators.

factor vs. surface area and scale factor vs. volume.

4[*]. **Explore problems and describe and confirm results using various representations.**

- Students in both vignettes explore problems and describe their results in various ways; the representations used are generally concrete and written.

8. **Recognize and apply unifying concepts and processes which are woven throughout mathematics.**

- The students in *Sketching Similarities* recognize and apply the concept of proportion. The students in *Rod Dogs* recognize and apply the concept of powers.

9. **Use the process of mathematical modeling in mathematics and other disciplines, and demonstrate understanding of its methodology, strengths, and limitations.**

- Students in both vignettes establish a mathematical model for their problem situations and use it to solve their problems. The students in *Rod Dogs*, in particular, also note the limitations of modeling concretely and the need to move to a more symbolic model.

10. **Apply mathematics in their daily lives and in career-based contexts.**

- The students in both vignettes apply mathematics to a realistic problem.

11. **Recognize situations in other disciplines in which mathematical models may be applicable, and apply appropriate models, mathematical reasoning, and problem solving to those situations.**

- Students might investigate how the 2-dimensional relationships in *Sketching Similarities* and the 3-dimensional results from *Rod Dogs* could be used in art, architecture, engineering, business, and the building trades.

Standard 4. All students will develop reasoning ability and will become self-reliant, independent mathematical thinkers.

Building upon knowledge and skills gained in the preceding grades, experiences in grades 7-8 will be such that all students:

2[*]. **Draw logical conclusions and make generalizations.**

- Students in both vignettes draw logical conclusions and make generalizations based on the data they collect.

3[*]. **Use models, known facts, properties, and relationships to explain their thinking.**

- Students in both vignettes use models to explain their thinking.

[*] Reference is made here to Indicators 2, 3, and 5, which are also listed for grade 4, since the Standards specify that students demonstrate continued progress in these indicators.

5[*]. Analyze mathematical situations by recognizing and using patterns and relationships.

- Students in both vignettes look for patterns to help them solve the problems.

6. Make conjectures based on observation and information, and test mathematical conjectures and arguments.

- Students in both vignettes make conjectures and test them.

7. Justify, in clear and organized form, answers and solution processes in a variety of problems.

- Students in both vignettes justify their answers in writing and orally.

8. Follow and construct logical arguments, and judge their validity.

- In both vignettes, the students could be challenged to explain why they thought that their generalizations would always work.

9. Recognize and use deductive and inductive reasoning in all areas of mathematics.

- The students in both vignettes use inductive reasoning to find a pattern and make a generalization.

10. Utilize mathematical reasoning skills in other disciplines and in their lives.

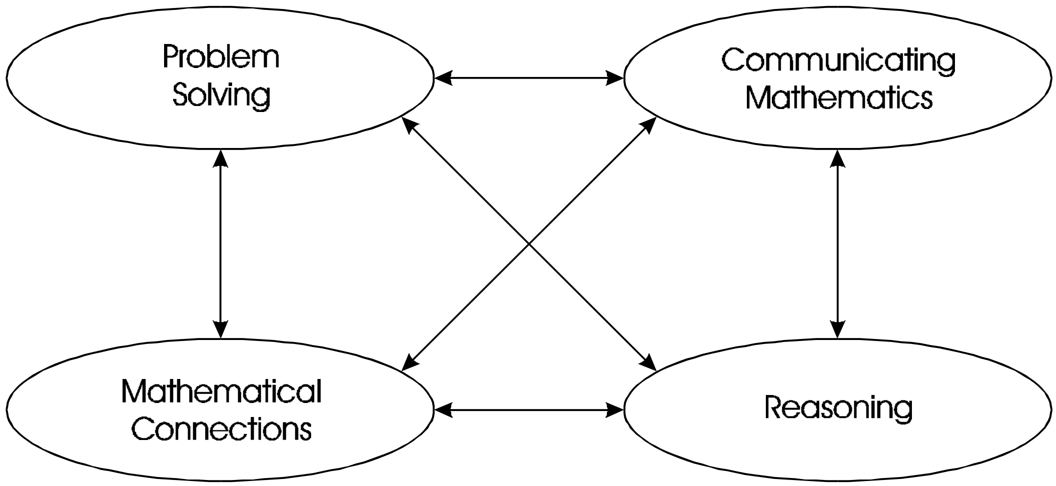
- Both vignettes illustrate the use of mathematical reasoning in real life.

11. Use reasoning rather than relying on an answer-key to check the correctness of solutions to problems.

- The students base their judgments about correctness of their results on the explanations given in the summary discussion rather than on an answer key.

[*] Reference is made here to Indicators 2, 3, and 5, which are also listed for grade 4, since the Standards specify that students demonstrate continued progress in these indicators.

The First Four Standards — Grades 9-12



Overview

New Jersey's *Mathematics Standards* calls for a shift in emphasis from a high school curriculum often dominated by memorization of isolated facts and procedures and by proficiency with paper-and-pencil skills to one that stresses understanding of concepts, multiple representations and connections, mathematical modeling, and mathematical problem solving.

The distinction between mathematical **problem solving** and doing mathematics should begin to blur in the high school grades. The problem-solving strategies learned in the earlier grades should have become increasingly internalized and integrated to form a broad basis for doing mathematics, regardless of the specific topic being addressed. From this perspective, problem solving is much more than solving word problems; it is the process by which mathematical ideas are constructed and reinforced. There is more emphasis in high school on introducing new mathematical concepts and tools as responses to problem situations in mathematics, and on developing students' ability to pose problems themselves.

Through extensive experiences with mathematical **communication**, students improve their understanding of mathematics. Students must be able to describe how they obtain an answer or the difficulties they encounter in trying to solve a problem. Facility with mathematical language enables students to form multiple representations of ideas, express relationships within and among these representations, and form generalizations.

High school students should continue to experience two types of **mathematical connections** — those within mathematics, and those to other areas. First, students should make connections between different mathematical representations of the same concept or process. Students who are able to apply and translate among different representations of the same problem situation or of the same mathematical concept have not only a powerful, flexible set of tools for solving problems but also a deeper appreciation of the consistency and beauty of mathematics. Unifying ideas within mathematics to be emphasized in the high school grades include mathematical modeling, variation (how a change in one thing is associated with a change in another), algorithmic thinking (developing, interpreting, and analyzing mathematical procedures), mathematical argumentation, and a continued focus on multiple representations.

Second, students in high school should regularly discuss the connections between mathematics and other subjects and the real world. Connections between mathematics and science are particularly plentiful. Examples of such activities are:

- Students use computer-aided design (CAD) to produce scale drawings or models of three-dimensional objects such as houses.
- Students use statistical techniques to predict and analyze election results.
- Cooling a cup of coffee provides an opportunity for students to collect and analyze data while learning about the process of cooling: *Why does cooling occur? Does the coffee cool at a constant rate? Does the temperature follow an asymptotically decreasing pattern?* By using temperature probes connected to a computer or graphing calculator (CBL), students can collect data on the temperature of a cup of hot coffee as it cools, plot this data on a coordinate graph, and describe the pattern verbally and with an equation.
- Students might explore models using logarithmic scales by studying the Richter scale used to measure the strength of earthquakes, the decibel scale for sound loudness, or the scale used to describe the brightness of stars.
- Students might develop experiments in which they collect data in order to investigate polynomials of higher degree. They can examine the collected data by using "finite differences" to predict the degree of the polynomial. In addition, this graphic approach helps students develop a better understanding of the concept of "zeros of a function."
- Students at the precalculus level might use science experiments to investigate trigonometry. Students enrolled in physics classes often use the Law of Sines and Law of Cosines prior to their development and discussion in precalculus classes. Thus, there is a need to reexamine the order of presentation of topics both in science and in mathematics.
- Trigonometric functions can be applied not only to modeling terrestrial and astronomical problems requiring indirect measurement but also to describing the motion of water waves, waves in a rope, sound waves, and light waves.
- Students can collect data to model rational functions, including not only $y = 1/x$ but also more complex equations such as $y = (x-1)(x+2)/(x+2)(x-3)$. Students should see why the data excludes the domain value of -2 as well as why there is an asymptote at $x=3$.
- Students might integrate the study of geometric sequences or logarithmic and exponential functions with science topics involving growth and decay. (See *Breaking the Mold* at the end of the Introduction of this *Framework* in which students look at the growth patterns of living things and the vignette involving carbon dating at the beginning of the Introduction.)
- Students might conduct experiments involving velocity with constant acceleration, such as dropping a ball, to study parabolas and quadratic equations.
- Students might study vectors in conjunction with complex numbers.
- Students can relate three-dimensional figures to the geometry of molecules, crystals, and symmetry.
- Students might link the study of solving linear equations in mathematics classes to the

balancing of equations in chemistry.

- The study of direct and inverse variation and different types of functions might be linked to the study of volcanic action and earthquakes.
- The study of calculus and physics might be integrated. In fact, a team-taught course might be more appropriate than the present approach, especially for the applications of calculus. This might prove more appropriate for those students who do not need a theoretical approach to calculus at this time.
- Students might design and construct a container that will sound an alarm when opened. In completing this task, they must use measurement, geometry, numerical operations, algebra, and mathematical reasoning as well as knowledge of electrical circuits, wiring, switches, electricity, insulators, conductors, Ohm's law and its use, and cost estimation.

A student who is doing mathematics often makes a conjecture by generalizing from a pattern of observations made in specific cases (inductive reasoning) and then tests the conjecture by constructing either a logical verification or a counterexample (deductive reasoning). High school students need to appreciate the role of both forms of **reasoning**. They should also learn that deductive reasoning is the method by which the validity of a mathematical assertion is finally established. Much inductive reasoning may take place in algebra, with students looking for patterns that arise in number sequences, making conjectures about general algebraic properties based on their observations, and verifying their conjectures with numerical substitutions. Students can be introduced to deductive reasoning by examining everyday situations, such as advertising, in which logic arises. Logical arguments in mathematical situations need not follow any specific format and may be presented orally or in writing in the student's own words.

High school students focus on mathematical problem solving, using multiple representations of mathematical concepts, mathematical connections, and reasoning throughout all of their mathematics learning. As they learn and do mathematics, they should regularly encounter situations where they are expected to discuss and solve problems, develop mathematical models, explain their results, and justify their reasoning. The content of these four standards is inextricably interwoven with the fabric of mathematics.

On-Line Resources

http://dimacs.rutgers.edu/nj_math_coalition/framework.html/

The *Framework* will be available at this site during Spring 1997. In time, we hope to post additional resources relating to this standard, such as grade-specific activities submitted by New Jersey teachers, and to provide a forum to discuss the *Mathematics Standards*.

The First Four Standards — Grade 9-12

Vignette — Making Rectangles

Standards: In addition to the First Four Standards, this vignette highlights Standards 7 (Spatial Sense), 11 (Patterns), and 13 (Algebra).

The problem: Yesterday, Mrs. Ellis' class finished a unit in which they used algebra tiles to help them develop procedures for multiplying binomials. Today, Mrs. Ellis began by asking the students to consider the following problem: *Suppose we have a collection of red x^2 tiles, orange x tiles, and yellow unit tiles. Can we put them together to form a rectangle? For example, if we have one red tile, five orange tiles, and four yellow tiles, we can make the rectangle at the right. What combinations of tiles can form a rectangle, and what combination cannot?*

	x	**1**	**1**	**1**	**1**
x	x^2	x	x	x	x
1	x	1	1	1	1

The discussion: Mrs. Ellis encouraged the students to share their ideas for how they might go about solving this problem. Some of their questions were: *What materials might be helpful in working on this problem? How could we use the algebra tiles? What might be some combinations of tiles that we might try? How could we keep track of what we have tried? Can we use more than one of the red x^2 tiles? Will we be able to try out all of the possible combinations? Do you suppose that we will find some sort of pattern that can help us predict which combinations will work and which will not?*

Solving the problem: The students worked in groups of three or four over a period of three days. They tried different combinations of tiles, recording how many of each tile they used, whether or not that combination could be used to form a rectangle, and, if so, the dimensions of the rectangle. As they worked, they began to notice patterns. Some of their comments were: *This seems like multiplying binomials but in reverse. When it works, the product of the number of 1s on the top times the number of 1s on the sides equals the number of yellow unit tiles. If there's just one of the red tiles, then the sum of the number of 1s on the top plus the number of 1s on the side equals the number of orange tiles.* Each group summarized its conclusions and the patterns they found in a report.

Summary: Mrs. Ellis asked the groups to exchange reports with another group, then read, review, and comment on the other group's report. Each group then had an opportunity to review the comments on their report. Mrs. Ellis asked the students about the patterns they had found. *Are there some patterns that both of your groups found? Are there others that only one of the groups found? She recorded the findings on the board. How can we be sure all of these statements are correct?* The students suggested that, if everyone agreed with a statement and could justify their reasoning, then it should be accepted as correct. They discussed each of the statements, explaining their reasoning and arguing about some of the statements. For homework, Mrs. Ellis asked the students to use their findings to make some predictions about other combinations of tiles and to relate their results to the idea of factoring binomials. Mrs. Ellis expects that in the next classes she will connect the problem of making a rectangle from a red x^2 tiles, b orange x tiles, and c yellow unit tiles to the problem of factoring $ax^2 + bx + c$.

The First Four Standards — Grades 9-12

Vignette — Ice Cones*

Standards: In addition to the First Four Standards, this vignette highlights Standards 6 (Number Sense), 7 (Geometry), 8 (Numerical Operations), 9 (Measurement), 11 (Patterns), 13 (Algebra), and 15 (Building Blocks of Calculus).

The problem: Ms. Longhart began class by posing the following problem for her students: *Suppose that you are setting up a water ice stand for the summer and are trying to decide how to make the cones in which you will serve the water ice. You've found some circles of radius 10 cm that are the right type of paper and have figured out that by cutting on a radius, you can make cones. You decide that you would like to make cones that hold as much water ice as possible, so you can charge a higher price. What will be the radius of the base of your cones? What will be the height?*

The discussion: Ms. Longhart asked the students what materials might be useful in helping them solve the problem. Some students suggested making models out of paper circles, while others thought that writing equations and using graphing calculators to find the maximum volume would be best. After some discussion of the relative merits of each approach, Ms. Longhart suggested that they do both and compare their answers.

Solving the problem: The students separated into groups to work on the problem. Most students remembered that the volume of a cone is $1/3\pi r^2 h$. They made a variety of paper cones out of circles with radius 10cm, and measured the radius and height of those cones. Using the formula, and a calculator, they generated a table of values, trying to find the maximum volume. They wanted to graph the formula using their calculator, but realized that they needed to solve for h in terms of r. After some initial difficulties, they decided that, since the original circles had a radius of 10 cm, the height of the resulting cone must be $\sqrt{100 - r^2}$. Then they graphed the equation they had generated on the graphing calculator and used the graph to find the maximum volume. Finally, each group summarized its findings in writing.

Summary: Each group listed its actual measurements and its results generated on the graphing calculator on the board and then explained any discrepancies that might have occurred. The class as a whole discussed the accuracy of the solutions. One of the students noticed that, in the cone of maximum volume, the radius was much larger than the height of the cone, and asked why that happened. Ms. Longhart asked the class to think about some possible reasons. To summarize the lesson, Ms. Longhart asked the students to list in their journals all of the mathematical concepts that they used to solve the problem in class that day. For homework, she asked the students to (1) describe how the function they generated would change if the radius of the circle was 8 cm, 9 cm, 11 cm, or 12 cm; (2) find the maximum volumes and corresponding radius for each of the new functions; and (3) determine whether there is a relationship between the radius and the maximum volume for each of the five functions.

* Adapted from Longhart, Karen. "Making Connections." *Eightysomething!* Volume 3, Number 2, Summer 1994.

The First Four Standards — Grades 9-12

Vignette — Building Parabolas

Standards: In addition to the First Four Standards, this vignette highlights Standards 7 (Geometry), 11 (Patterns), and 13 (Algebra).

The problem: Before this session, students in Mr. Evans' class investigated different situations that can be modeled using quadratic functions. They looked at how to maximize the area of a yard given a fixed amount of fencing, and how to predict the path of a rocket. They graphed many quadratic functions, some by plotting points and some on the graphing calculator, and discovered that all of the quadratic functions have graphs that are parabolas. For this session, they went to the computer lab to investigate the relationship of each of the constants in the general form of the parabola to the graph of that equation.

The discussion: Before beginning work on the computers, the students reviewed the general shape of a parabola and discussed the differences between one quadratic function and another: the width of the parabola, how high up or down the vertex is, and whether the parabola opens up or down. Mr. Evans introduced the general form of the equation of a parabola, $y = a(x - h)^2 + k$, and asked the students to predict how each of a, h, and k will affect the graph of the parabola. He asked them to explain their reasoning and record their predictions individually in their notebooks, and then he led a discussion of their predictions.

Solving the problem: Each pair of students used a program on *Green Globs* software to predict the equation for a series of parabolas, keeping notes on how the different constants seemed to affect the graphs. There was considerable excitement in the room, as well as some disagreements and some frustration at times. Some pairs of students found that several of the graphs required many attempts before the correct equation was found. At the conclusion of the computer activity, the students compared their results to their predictions and discussed their findings with each other in pairs. They then individually wrote a description of how the values of a, h, and k in the general equation $y = a(x - h)^2 + k$ affect the graph of $y = x^2$.

Summary: For homework, Mr. Evans asked the students to use their findings to sketch the graphs of several parabolas without using graphing calculators and then to check their sketches by using graphing calculators. He suggested that they revise their journal entries if they find that some of their hypotheses don't work. Mr. Evans began class the next day by having pairs of students play the computer game *Green Globs*, allowing the students to use only parabolas to hit the globs on the coordinate grid. After about 15 minutes, he led a discussion of the students' findings about parabolas, asking them how they arrived at their hypotheses, what steps they took to verify them, and whether they modified their hypotheses based on their experiences with the homework and the computer game. He then asked the students to think of other areas of mathematics that seem to be related, making connections between their findings about parabolas and what they learned last year about geometric transformations.

The First Four Standards — Grades 9-12

Indicators

The cumulative progress indicators for grade 12 for each of the First Four Standards appear in boldface type below the standard. Each indicator is followed by a brief discussion of how the preceding grade level vignettes might address the indicator in the classroom in grades 9-12. The Introduction to this *Framework* contains three vignettes describing lessons for grades 9-12 which also illustrate the indicators for the First Four Standards; these are entitled *On the Boardwalk*, *A Sure Thing!?*, and *Breaking the Mold*.

> Standard 1. All students will develop the ability to pose and solve mathematical problems in mathematics, other disciplines, and everyday experiences.

Building upon knowledge and skills gained in the preceding grades, experiences in grades 9-12 will be such that all students:

4[*]. Pose, explore, and solve a variety of problems, including non-routine problems and open-ended problems with several solutions and/or solution strategies.

- In *Making Rectangles*, the students explore and solve an open-ended question. In *Ice Cones*, the students explore and solve a problem using several different solution strategies (concrete materials or writing equations). In *Building Parabolas*, the students explore an open-ended question.

5[*]. Construct, explain, justify, and apply a variety of problem-solving strategies in both cooperative and independent learning environments.

- In all three vignettes, the students work in small groups to construct, explain, and justify their strategies.

6[*]. Verify the correctness and reasonableness of results and interpret them in the context of the problems being solved.

- In all three situations, the students verify their results by sharing them and interpreting them in a whole-class discussion.

7[*]. Know when to select and how to use grade-appropriate mathematical tools and methods (including manipulatives, calculators and computers, as well as mental math and paper-and-pencil techniques) as a natural and routine part of the problem-solving process.

[*] Reference is made here to Indicators 4, 5, 6, 7, 8, 12, and 14 which are also listed for grade 8, since the Standards specify that students demonstrate continued progress in these indicators.

- Students in *Making Rectangles* use manipulatives to help them solve their problem. Some students in *Investigating Cones* choose to use manipulatives, while others use graphing calculators to help them solve the problem. Students in *Building Parabolas* use computers to help develop their ideas.

8[*]. Determine, collect, organize, and analyze data needed to solve problems.

- The students in *Making Rectangles* organize and analyze their results. Some of the students in *Ice Cones* decide to approach the problem by looking at some specific cones and finding their volume. (Note that this was not a particularly efficient approach to solving this problem.) The students in *Building Parabolas* decide which parabolas to try out in playing *Green Globs* and then collect, organize, and analyze their data.

12[*]. Construct and use concrete, pictorial, symbolic, and graphical models to represent problem situations and effectively apply processes of mathematical modeling in mathematics and other areas.

- Students in *Making Rectangles* use concrete, pictorial, and symbolic models to represent their problem situation. Students in *Ice Cones* use concrete, symbolic, and graphical models to represent the problem and develop a mathematical model to describe the height of the desired cone. Students in *Building Parabolas* use symbolic and graphical models to represent the problem situation.

14[*]. Persevere in developing alternative problem-solving strategies if initially selected approaches do not work.

- The students who experience initial difficulties in *Ice Cones* are often the ones who try a "guess-and-check" strategy. They decide to try a different approach, using an equation, when they encounter problems. The students in *Building Parabolas* must use alternative approaches to predict the correct equations from the *Green Globs* program. There is considerable frustration for some students and perseverance is required as they make numerous attempts to find the correct function.

15. Use discovery-oriented, inquiry-based, and problem-centered approaches to investigate and understand the mathematical content appropriate to the high-school grades.

- The students in *Making Rectangles* are learning about factoring by solving a problem. The students in *Ice Cones* are exploring how to find a maximum in a problem involving both algebra and geometry. In *Building Parabolas*, the students are using a question about parabolas to learn about how the coefficients of the equation affect the graph.

16. Recognize, formulate, and solve problems arising from mathematical situations, everyday experiences, applications to other disciplines, and career applications.

- The problem in *Ice Cones* arises from career applications. The problems in *Building Parabolas* and *Making Rectangles* arise from mathematics itself.

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[*] Reference is made here to Indicators 4, 5, 6, 7, 8, 12, and 14 which are also listed for grade 8, since the Standards specify that students demonstrate continued progress in these indicators.

17. Monitor their own progress toward problem solutions.

- The students in *Making Rectangles* keep a record of the different combinations of tiles they try, whether or not the combination could be used to form a rectangle, and the dimensions of the rectangle; they begin to notice patterns. The students in *Ice Cones* try a different strategy when they find that they are not making progress. In *Building Parabolas* the very use of the computer game *Green Globs*, helps students monitor their own progress towards a solution.

18. Explore the validity and efficiency of various problem-posing and problem-solving strategies, and develop alternative strategies and generalizations as needed.

- The students in all three vignettes share the strategies they use to solve their problems. In *Tiling a Floor*, the students select a variety of materials, including a computer program, to solve the problem. In *Sharing Cookies*, students use different methods to solve the same problem.

Standard 2. All students will communicate mathematically through written, oral, symbolic, and visual forms of expression.

Building upon knowledge and skills gained in the preceding grades, experiences in grades 9-12 will be such that all students:

1*. Discuss, listen, represent, read, and write as vital activities in their learning and use of mathematics.

- Students in all three vignettes use a variety of communication activities: listening, discussing in small and large groups, representing in algebraic and graphical or concrete contexts, and reading and writing about their solutions.

2*. Identify and explain key mathematical concepts, and model situations using oral, written, concrete, pictorial, and graphical methods.

- In *Ice Cones*, the teacher asks the students to list all of the mathematical concepts they use in solving the problem in their journals. In all three vignettes, students model situations using different methods: oral, written, concrete, and graphical.

3*. Represent and communicate mathematical ideas through use of learning tools such as calculators, computers, and manipulatives.

- Some students in *Ice Cones* use graphing calculators; others use manipulatives (paper circles). The students in *Building Parabolas* use a computer program. Students in *Making Rectangles* use manipulatives.

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* Reference is made here to Indicators 1, 2, 3, 4, 5, 6, 7, and 8, which are also listed for grade 8, since the Standards specify that students demonstrate continued progress in these indicators.

4*. **Engage in mathematical brainstorming and discussions by asking questions, making conjectures, and suggesting strategies for solving problems.**

- In all three vignettes, the students engage in brainstorming before solving the problem.

5*. **Explain their own mathematical work to others, and justify their reasoning and conclusions.**

- In all three vignettes, the students explain their work to the others in the class.

6*. **Identify and explain key mathematical concepts and model situations using geometric and algebraic methods.**

- In *Making Rectangles*, most of the students draw pictures of the various tiles they can make and write about the tiles using algebraic notation. For example, they write that they can use one red x^2 tile, two orange x tiles, and one yellow unit tile to make a larger square tile and record this as $x^2 + 2x + 1 = (x +1)(x +1)$. The students in *Ice Cones* model the situation using both geometric (paper circles) and algebraic (graphing calculator) methods. In *Building Parabolas*, the students model the situation both algebraically with equations and geometrically with graphs.

7*. **Use mathematical language and symbols to represent problem situations, and recognize the economy and power of mathematical symbolism and its role in the development of mathematics.**

- The students in all three vignettes use mathematical language and symbols to represent the problem situation.

8*. **Analyze, evaluate, and explain mathematical arguments and conclusions presented by others.**

- The students in all three vignettes present their results to the class, evaluating and explaining their conclusions.

9. **Formulate questions, conjectures, and generalizations about data, information, and problem situations.**

- In *Making Rectangles*, the students use their findings to make predictions about combinations of tiles. In *Ice Cones*, the students make conjectures and generalizations, including determining whether there is a general relationship between the radius and the maximum volume for each of five different cones. The students in *Building Parabolas* make conjectures and generalizations about the effects of the coefficients in the general form of the equation on the graph.

10. **Reflect on and clarify their thinking so as to present convincing arguments for their conclusions.**

- Students in all three vignettes are asked to reflect on and clarify their thinking by sharing with others in small groups and by summarizing their findings in writing.

* Reference is made here to Indicators 1, 2, 3, 4, 5, 6, 7, and 8, which are also listed for grade 8, since the Standards specify that students demonstrate continued progress in these indicators.

Standard 3. All students will connect mathematics to other learning by understanding the interrelationships of mathematical ideas and the roles that mathematics and mathematical modeling play in other disciplines and in life.

Building upon knowledge and skills gained in the preceding grades, experiences in grades 9-12 will be such that all students:

1*. View mathematics as an integrated whole rather than as a series of disconnected topics and rules.

- In *Making Rectangles*, factoring binomials is related to area and multiplication. In *Ice Cones*, the students are drawing from concepts in algebra, geometry, and calculus. In *Building Parabolas*, the students have already investigated different situations that can be modeled by parabolas. At the end, they also relate their work to previous work with geometric transformations.

2*. Relate mathematical procedures to their underlying concepts.

- *Making Rectangles* relates factoring to the underlying concepts of area and multiplication. *Ice Cones* relates finding a maximum to the underlying concepts of volume and solving equations. *Building Parabolas* relates the shape of a graph to its equation.

3*. Use models, calculators, and other mathematical tools to demonstrate the connections among various equivalent graphical, concrete, and verbal representations of mathematical concepts.

- Students in *Making Rectangles* use their models to demonstrate the connection between the geometric topic of area and the algebraic topic of factoring. The students in *Ice Cones* use calculators to help demonstrate the connection between the equation and the maximum value for the volume. Students in *Building Parabolas* use computers to demonstrate the connection between the algebraic and graphical representations for parabolas.

8*. Recognize and apply unifying concepts and processes which are woven throughout mathematics.

- Students in *Making Rectangles* and *Building Parabolas* are focusing on multiple representations. Students in *Ice Cones* are applying mathematical modeling to a real-life situation.

9*. Use the process of mathematical modeling in mathematics and other disciplines, and demonstrate understanding of its methodology, strengths, and limitations.

- The *Ice Cones* vignette illustrates the process of mathematical modeling in real life. In *Making Rectangles* and *Building Parabolas*, students use mathematical modeling to discover patterns which ultimately help them to solve the problems.

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* Reference is made here to Indicators 1, 2, 3, 8, 9, 10, and 11, which are also listed for grade 8, since the Standards specify that students demonstrate continued progress in these indicators.

10[*]. Apply mathematics in their daily lives and in career-based contexts.

- *Making Rectangles* and *Ice Cones* involve applying mathematics in career-based contexts.

11[*]. Recognize situations in other disciplines in which mathematical models may be applicable, and apply appropriate models, mathematical reasoning, and problem solving to those situations.

- All three vignettes focus on modeling within mathematics. The 9-12 Overview describes a number of situations in other disciplines in which mathematical models are applicable.

12. Recognize how mathematics responds to the changing needs of society, through the study of the history of mathematics.

- The problems discussed in the vignettes are not presented in a social or historical context. However, students can also investigate the role of the quadratic functions, discussed in *Building Parabolas*, in ballistics, and can extend their *Ice Cones* discussions to include other examples of packaging.

> Standard 4. All students will develop reasoning ability and will become self-reliant, independent mathematical thinkers.

Building upon knowledge and skills gained in the preceding grades, experiences in grades 9-12 will be such that all students:

2[*]. Draw logical conclusions and make generalizations.

- Students in all three vignettes draw logical conclusions and make generalizations.

5[*]. Analyze mathematical situations by recognizing and using patterns and relationships.

- Students in all three vignettes look for patterns and relationships in order to make their generalizations.

8[*]. Follow and construct logical arguments, and judge their validity.

- In *Making Rectangles*, the students discuss how they will know that their work is correct. They judge the validity of each others' arguments in the subsequent discussion. In *Ice Cones*, the students worked in groups and explained their results and any discrepancies they encountered. The class as a whole discussed the accuracy of solutions. In *Building Parabolas*, also in a class discussion, students explained their hypotheses, how they verified them, and whether they had to modify them based on their experiences.

9[*]. Recognize and use deductive and inductive reasoning in all areas of mathematics.

- All three vignettes deal primarily with inductive reasoning. However, building on the activities in these vignettes, students can use deductive reasoning to show the effect of

[*] Reference is made here to Indicators 2, 5, 8, 9, 10, and 11, which are also listed for grade 8, since the Standards specify that students demonstrate continued progress in these indicators.

increasing k by 3 in *Building Parabolas* or to prove that certain collections of titles can or cannot form rectangles in *Making Rectangles*.

10[*]. Utilize mathematical reasoning skills in other disciplines and in their lives.

- *Making Rectangles* and *Ice Cones* illustrate the use of mathematics in daily life.

11[*]. Use reasoning rather than relying on an answer-key to check the correctness of solutions to problems.

- None of the students in any of the vignettes checks answers with an answer key; they all report their answers to the class and explain how they got them.

12. Make conjectures based on observation and information, and test mathematical conjectures, arguments, and proofs.

- The students in *Making Rectangles* make some initial conjectures about which tiles they can make and then test them. The students in *Ice Cones* make and test conjectures in their homework as they begin to generalize their results. The students in *Building Parabolas* make conjectures about the effects of each constant and then test these using the computer.

13. Formulate counter-examples to disprove an argument.

- This indicator is not explicitly addressed in these vignettes. However, in the discussion in any of the vignettes, it is possible that some of the analysis of solutions will lead students to present counterexamples to disprove another student's argument.

[*] Reference is made here to Indicators 2, 5, 8, 9, 10, and 11, which are also listed for grade 8, since the Standards specify that students demonstrate continued progress in these indicators.